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A CENTRAL LIMIT THEOREM FOR THE OVERLAP IN THE HOPFIELD MODEL¹

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We consider the Hopfield model with n neurons and an increasing number $p = p(n)$ of randomly chosen patterns. Under the condition $(p^3 \log p)/n \rightarrow 0$, we prove for every fixed choice of overlap parameters a central limit theorem as $n \rightarrow \infty$, which holds for almost all realizations of the random patterns. In the special case where the temperature is above the critical one and there is no external magnetic field, the condition $(p^2 \log p)/n \rightarrow 0$ suffices. As in the case of a finite number of patterns, the central limit theorem requires a centering which depends on the random patterns.

1. Introduction. The so-called Hopfield model was introduced as a model for an associative memory in the context of an artificial neural network with $n \in \mathbb{N}$ neurons. Roughly speaking, it deals with the problem of recognizing one out of $p \in \mathbb{N}$ stored patterns; for details, see [9], for example. We denote by $\xi^\mu = (\xi_1^\mu, \dots, \xi_n^\mu) \in \{-1, 1\}^n$ the codification of the μ th stored pattern, $\mu \in \{1, \dots, p\}$, and by $\sigma_i \in \{-1, 1\}$ the neural activity of the i th neuron, $i \in \{1, \dots, n\}$. Following Hopfield [10], we can describe the model as a spin system with the Hamiltonian

$$(1.1) \quad H_n(\sigma, \xi) = -\frac{1}{2n} \sum_{\mu=1}^p \sum_{i,j=1}^n \xi_i^\mu \xi_j^\mu \sigma_i \sigma_j, \quad n \in \mathbb{N},$$

and the corresponding Glauber single-spin dynamics at zero temperature on the set of spin configurations $\{-1, 1\}^n$. The spin system is said to retrieve the stored pattern ξ^μ when the spin configuration $(\sigma_1, \dots, \sigma_n) \in \{-1, 1\}^n$ converges under the Glauber dynamics to ξ^μ , respectively, $-\xi^\mu$ for some $\mu \in \{1, \dots, p\}$.

In [1] generalized Glauber single-spin dynamics at finite temperature $1/\beta \in (0, \infty)$ are introduced on the set of spin configurations, again governed by the Hopfield Hamiltonian (1.1). These dynamics describe a reversible, irreducible Markov process which converges to its equilibrium distribution. The equilibrium distribution is the finite-volume Gibbs measure $P_{n, \beta, \xi}$ given by

$$(1.2) \quad dP_{n, \beta, \xi} = \exp\{-\beta H_n(\sigma, \xi)\} d\mathbb{P}_\sigma / Z_{n, \beta, \xi},$$

where $\mathbb{P}_\sigma = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes \mathbb{N}}$ and the partition function $Z_{n, \beta, \xi}$ is the appropriate normalization.

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From now on, we allow p to depend on n in a nondecreasing way satisfying $p \leq n$. We will consider $(\sigma_i)_{i \in \mathbb{N}}$ and $(\xi_i^\mu)_{i \in \mathbb{N}}$ with $\mu \in \mathbb{N}$ as random variables and assume that the family $\{\sigma_i, \xi_j^\mu \mid i, j, \mu \in \mathbb{N}\}$ is independent with $\mathbb{P}(\sigma_i = \pm 1) = 1/2$ and $\mathbb{P}(\xi_j^\mu = \pm 1) = 1/2$. We denote by $\mathbb{P}_\xi = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes \mathbb{N}^2}$ the marginal distribution of $\xi = (\xi_i^\mu)_{i, \mu \in \mathbb{N}}$, and similarly, by $\mathbb{P}_\sigma = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes \mathbb{N}}$ the marginal distribution of $\sigma = (\sigma_i)_{i \in \mathbb{N}}$.

In this setting, the Hopfield model may be regarded as an example of a mean-field spin system with random interaction. It has been studied widely; see [2] and the references given there. Note that already in 1977, Figotin and Pastur [6, 7] introduced and discussed a class of spin models which included the one with the Hopfield Hamiltonian (1.1).

For $i \in \{1, \dots, n\}$ we denote by $\xi_i = (\xi_i^\mu)_{\mu \in \{1, \dots, p\}}$ the vector consisting of the i th components of the first p patterns. Keep in mind that $\xi_i \in \mathbb{R}^p$ depends on n via the dimension p . In the sequel, we will investigate the behavior of the so-called overlap

$$(1.3) \quad \frac{1}{n} S_n(\sigma, \xi) = \frac{1}{n} \sum_{i=1}^n \xi_i \sigma_i \in \mathbb{R}^p$$

under the equilibrium distribution $P_{n, \beta, \xi}$ as $n \rightarrow \infty$. The overlap compares the spin configuration σ to the stored patterns ξ^μ , $\mu \in \{1, \dots, p\}$, in the sense that the μ th component of the overlap (1.3)—called the μ th overlap parameter—equals 1 if and only if $\sigma_i = \xi_i^\mu$ for all $i \in \{1, \dots, n\}$. The Hamiltonian (1.1) can be rewritten as a quadratic function of the overlap which provides the following convenient notation:

$$H_n(\sigma, \xi) = -\frac{n}{2} \left\| \frac{1}{n} S_n(\sigma, \xi) \right\|^2.$$

For simplicity, we will drop the explicit dependence on σ and ξ whenever no confusion may arise. So we will write S_n instead of $S_n(\sigma, \xi)$, for instance.

For $p = 1$ the Hopfield model is equivalent to the Curie–Weiss model. Let us fix some notation concerning the latter. For $\beta > 0$, denote by $z^\pm(\beta)$ the largest (resp. smallest) solution $z \in (-1, 1)$ of the Curie–Weiss equation $\beta z = \operatorname{arctanh} z$. Note that $z^+(\beta) = -z^-(\beta) \neq 0$ for β above the critical inverse temperature $\beta_c = 1$, while $z^\pm(\beta) = 0$ for $\beta \leq \beta_c$. The free energy in the Curie–Weiss model equals

$$f_{\text{CW}}(\beta) = \frac{\beta}{2} z^\pm(\beta)^2 - I(z^\pm(\beta)),$$

where

$$I(z) = \begin{cases} \frac{1}{2}(1+z)\log(1+z) + \frac{1}{2}(1-z)\log(1-z), & \text{for } |z| \leq 1, \\ \infty, & \text{otherwise,} \end{cases}$$

denotes the rate function which governs the large deviations of the spins per site in the Curie–Weiss model; compare [5], Section IV.4. To extend the definition of $z^\pm(\beta)$ to the case of a so-called external magnetic field of strength

$h \neq 0$, let $z(\beta, h)$ denote the unique solution $z \in (-1, 1)$ of $\beta z + h = \operatorname{arctanh} z$ which satisfies $\operatorname{sign} z = \operatorname{sign} h$. In addition, we extend the notation $f_{\text{CW}}(\beta)$ for the free energy in the Curie–Weiss model by defining

$$f_{\text{CW}}(\beta, h) = \begin{cases} \frac{\beta}{2} z(\beta, h)^2 + h z(\beta, h) - I(z(\beta, h)), & \text{if } h \neq 0, \\ f_{\text{CW}}(\beta), & \text{otherwise.} \end{cases}$$

Note that

$$(1.4) \quad f_{\text{CW}}(\beta, h) = -\frac{1}{2\beta} (\operatorname{arctanh} z - h)^2 + \log \cosh \operatorname{arctanh} z,$$

where $z = z^\pm(\beta)$ in the case $h = 0$ and $z = z(\beta, h)$, otherwise.

For investigating the behavior of the overlap (1.3), we also extend the notion of the Gibbs measure $P_{n, \beta, \xi}$ given in (1.2) to the case of an external magnetic field he_l with strength $h \neq 0$ in the direction of the l th unit vector e_l in \mathbb{R}^p . Let

$$(1.5) \quad dP_{n, \beta, he_l, \xi} = \exp\{-\beta H_n + \langle S_n, he_l \rangle\} d\mathbb{P}_\sigma / Z_{n, \beta, he_l, \xi},$$

with the appropriate normalization $Z_{n, \beta, he_l, \xi}$. In [2] it is shown that under the assumption $p/n \rightarrow 0$, for \mathbb{P}_ξ -almost all ξ , the distribution of the overlap S_n/n under $P_{n, \beta, he_l, \xi}$ converges weakly, as $n \rightarrow \infty$ and then $h \rightarrow 0^\pm$, to the Dirac measure concentrated at $z^\pm(\beta)e_l$. Recall that $z^\pm(\beta) = 0$ for $\beta \leq \beta_c = 1$, which implies uniqueness of the limiting measure in this case. Let us remark that under the assumption $p/n \rightarrow 0$, in [3] a large deviation principle is proved for the distribution of every finite choice of overlap parameters under the Gibbs measure (1.2).

Considering the above-mentioned weak convergence, it is natural to ask for the fluctuations of the overlap around $z^\pm(\beta)e_l$. After some necessary preparations, we will state our main result, a central limit theorem for the overlap, in Section 2. In addition, we describe the asymptotic behavior of the partition functions in (1.2) and (1.5). In Section 3, we give the proofs of these results, based on a lemma that describes the asymptotic behavior of certain integrals. This key lemma itself is proved in Section 4 via an adaptation of Laplace's method.

2. Statement of results. To discuss the fluctuations of the overlap, we first choose a preferred pattern. This can be done in two different ways. In Theorem 2.5, we consider the unbiased Hamiltonian (1.1) and investigate the fluctuations under the condition that the overlap is already in a neighborhood of $z^\pm(\beta)e_l$. Alternatively, the preferred pattern can be chosen by introducing an external magnetic field as in (1.5). The latter case is treated in Theorem 2.7. From now on, we will make the the following assumption.

GENERAL ASSUMPTION. Let $p = p(n)$ be a nondecreasing function of n with $p \leq n$ for all $n \in \mathbb{N}$ and $\alpha = \alpha(n) = p/n \rightarrow 0$ as $n \rightarrow \infty$.

Before stating the central limit theorem, we need to have a closer look at its centering. In the case of the unbiased Hamiltonian (1.1) with $\beta < \beta_c$, the central limit theorem holds with center zero. Otherwise, even in the case of bounded p , the central limit theorem for $(\beta, h) \neq (\beta_c, 0)$ requires a ξ -dependent adjustment of the deterministic centering, where the deterministic one is suggested by the weak convergence of the overlap's distribution toward a Dirac measure. In [8], where the case of fixed p is treated, a linear adjustment of the centering was chosen. Here we will choose the optimal centering instead of a linear approximation.

To describe the appropriate centering, we have to control the influence of the random patterns. This can be done via the $(p \times p)$ matrix $(1/n) \sum_{i=1}^n \xi_i \xi_i^T$, which will play a key role. Under the assumption $p/n \rightarrow 0$ this matrix approaches the $(p \times p)$ identity matrix as $n \rightarrow \infty$ in the sense that the operator norm of

$$\Sigma^n(\xi) = \frac{1}{n} \sum_{i=1}^n \xi_i \xi_i^T - \text{Id}_{\mathbb{R}^p}$$

converges to zero for \mathbb{P}_ξ -almost all ξ . This is the assertion of the following proposition, which is a modification of [2], Theorem 2.4 and mainly follows from the proof of that theorem; see Section 3.

PROPOSITION 2.1. *Fix $\varepsilon > 0$ and define*

$$\alpha^* = \frac{1}{n} \max \left\{ p, \left(\frac{3 \log n}{\log(1 + \varepsilon)} \right)^4 \right\} \quad \text{and} \quad \varepsilon_n = \sqrt{\alpha^*} (2 + \sqrt{\alpha^*}) (1 + \varepsilon).$$

Then there exists an $\tilde{n}_0 \in \mathbb{N}$ such that

$$(2.1) \quad \mathbb{P}_\xi(\|\Sigma^n(\xi)\| > \varepsilon_n) \leq \frac{1}{n^2}$$

for all $n \geq \tilde{n}_0$. Furthermore, for \mathbb{P}_ξ -almost all ξ , there exists an $n_0(\xi) \geq \tilde{n}_0$ such that

$$(2.2) \quad \|\Sigma^n(\xi)\| \leq \varepsilon_n$$

for all $n \geq n_0(\xi)$.

REMARK 2.2. In the case of bounded p , using the law of the iterated logarithm, the bound ε_n in (2.2) can be improved to $\sqrt{\alpha}(2 \log \log n)^{1/2} (1 + \varepsilon)$; compare [8].

Let us fix some more notation. We denote by e_l the l th unit vector in \mathbb{R}^p . In addition, we set $e_{-l} = -e_l$ for convenience. The following index set L will be used to describe those directions the overlap favors under the equilibrium measure. This index set depends on the dimension p , on the inverse temperature $\beta > 0$, the presence ($h \neq 0$) or absence ($h = 0$) of an external magnetic

field and its direction e_l . Set

$$L = \begin{cases} \{\text{sign}(h)l\}, & \text{in the case } h \neq 0, \\ \{1\}, & \text{in the case } 0 < \beta < \beta_c \text{ and } h = 0, \\ \{-p, \dots, -1, 1, \dots, p\}, & \text{in the case } \beta > \beta_c \text{ and } h = 0. \end{cases}$$

We do not need to define the index set L for $(\beta, h) = (\beta_c, 0)$ because the central limit theorem fails in this case; compare Remark 2.6(b). The following proposition is a first step toward defining the centering for the central limit theorem.

PROPOSITION 2.3. *Let $\beta > 0$ and $h \geq 0$ be such that $(\beta, h) \neq (\beta_c, 0)$. As an abbreviation, we set $z = z^+(\beta)$ in the case $h = 0$, and $z = z(\beta, h)$, otherwise. Let $l \in \{-p, \dots, -1, 1, \dots, p\}$. For $\lambda \in \mathbb{R}^p$ define the ξ -dependent function*

$$(2.3) \quad \Phi(\lambda) = -\frac{1}{2\beta} \|\lambda - h e_l\|^2 + \frac{1}{n} \sum_{i=1}^n \log \cosh \langle \lambda, \xi_i \rangle.$$

Then, for all strictly positive $c_1 < (1 - \beta(1 - z^2))/\beta$, there exists an $r_1 > 0$, depending on β, h and c_1 only, and for \mathbb{P}_ξ -almost all ξ , there exists an $n_1(\xi) \geq n_0(\xi)$, which does not depend on the choice of l , such that for all $n \geq n_1(\xi)$ the following assertions hold:

(a) *For all λ in the closed ball $\overline{B_{r_1}(\text{arctanh } z e_l)}$, the matrix $-D^2\Phi(\lambda)$ is uniformly positive definite in the sense that*

$$\langle u, -D^2\Phi(\lambda)u \rangle \geq c_1 \|u\|^2 \quad \text{for all } u \in \mathbb{R}^p.$$

(b) *On the set $\overline{B_{r_1}(\text{arctanh } z e_l)}$, the map Φ has a unique maximum which is attained in a point $\lambda_l^n(\xi)$ satisfying*

$$(2.4) \quad \|\lambda_l^n(\xi) - \text{arctanh } z e_l\| \leq c_2 \varepsilon_n,$$

with $c_2 = 2|z|/c_1$. In particular, $\lambda_l^n(\xi) = 0$ in the case $\beta < \beta_c$ and $h = 0$.

REMARK 2.4. In [4], these assertions have been proved for $h = 0$ and p increasing at least linearly with $\log n$. The proof naturally extends to the case of small p and $h > 0$ and will be given in Section 3. If n is a multiple of 2^p and the realization ξ_1, \dots, ξ_n takes all possible values with the same frequency, then $\lambda_l^n(\xi) = \text{arctanh } z e_l$. Furthermore, due to (1.4), we always have $\Phi(\lambda_l^n(\xi)) \geq f_{\text{CW}}(\beta, h)$.

From now on, we fix

$$(2.5) \quad c_1 = \frac{1}{2\beta} (1 - \beta(1 - z^2))$$

for convenience. With the help of $\lambda_l^n(\xi)$, the appropriate centering $x_l^n(\xi)$ can be described by

$$x_l^n(\xi) = \frac{1}{\beta}(\lambda_l^n(\xi) - h e_l)$$

for $l \in \{-p, \dots, -1, 1, \dots, p\}$. Since $\lambda_l^n(\xi)$ is only defined for \mathbb{P}_ξ -almost all ξ and $n \geq n_1(\xi)$, we extend the definition of $x_l^n(\xi)$ by $x_l^n(\xi) = \beta^{-1}(\operatorname{arctanh} z - h)e_l = z e_l$ whenever $\lambda_l^n(\xi)$ is not defined. Keep in mind that $x_l^n(\xi)$ depends on β and h although this is not indicated by the notation. By (2.4) and the Curie–Weiss equation, we always have

$$\|x_l^n(\xi) - z e_l\| \leq \frac{1}{\beta} c_2 \varepsilon_n.$$

Now we are ready to state our main result, a \mathbb{P}_ξ -almost sure central limit theorem for the overlap. Let $\pi_k: \mathbb{R}^p \rightarrow \mathbb{R}^k$ (with $k \leq p$) denote the canonical projection.

THEOREM 2.5. *Let $\beta > 0$, $\beta \neq \beta_c$, $h = 0$ and let $k \in \mathbb{N}$ satisfy $k \leq p$ in the case of bounded p . Fix $\varepsilon \in (0, z^+(\beta)/8)$ in the case $\beta > \beta_c$ and $\varepsilon > 0$ arbitrary, otherwise. Suppose that $(p^2 \log p)/n \rightarrow 0$ in the case $\beta < \beta_c$ and $(p^3 \log p)/n \rightarrow 0$, otherwise. Then, for every $l \in L$ and \mathbb{P}_ξ -almost all ξ , the measures*

$$(2.6) \quad P_{n, \beta, \xi} \left(\sqrt{n} \pi_k \left[\frac{S_n}{n} - x_l^n(\xi) \right] \in \cdot \mid \left\| \frac{S_n}{n} - z^+(\beta) e_l \right\| < \varepsilon \right)$$

converge weakly, as n tends to infinity, to the Gaussian distribution on \mathbb{R}^k with mean zero and covariance matrix

$$C_\beta = \frac{1 - z^+(\beta)^2}{1 - \beta(1 - z^+(\beta)^2)} \operatorname{Id}_{\mathbb{R}^k}.$$

REMARK 2.6. (a) The condition $\varepsilon < z^+(\beta)/8$ in the case $\beta > \beta_c$ especially assures that the balls of radius ε with centers $z^+(\beta)e_\nu$, $\nu \in L$, are disjoint.

(b) Recall that the fluctuations of the spin per site in the Curie–Weiss model, which corresponds to a single component of the overlap in the Hopfield model, are non-Gaussian at the critical inverse temperature $\beta_c = 1$ (cf. [5], Theorem V.9.5).

(c) For $\beta < \beta_c$, the theorem reduces to

$$P_{n, \beta, \xi} \left(\pi_k \left[\frac{S_n}{\sqrt{n}} \right] \in \cdot \mid \left\| \frac{S_n}{n} \right\| < \varepsilon \right) \Rightarrow \mathcal{N} \left(0, \frac{1}{1 - \beta} \operatorname{Id}_{\mathbb{R}^k} \right),$$

\mathbb{P}_ξ -almost surely, which yields

$$P_{n, \beta, \xi} \left(\pi_k \left[\frac{S_n}{\sqrt{n}} \right] \right)^{-1} \Rightarrow \mathcal{N} \left(0, \frac{1}{1 - \beta} \operatorname{Id}_{\mathbb{R}^k} \right)$$

as an immediate consequence of Remark 2.10(b).

(d) By the triangle inequality, it is obvious that the conditioning in (2.6) may be replaced by $\|S_n/n - x_l^n(\xi)\| < \varepsilon$. In addition, it is intuitively clear that the conditioning can as well be replaced by $\|\pi_k[S_n/n - z^+(\beta)e_l]\| < \varepsilon$ or $\|\pi_k[S_n/n - x_l^n(\xi)]\| < \varepsilon$, respectively. Indeed, this can be shown along the lines of the proof of the central limit theorem given in Section 3.

Selecting the preferred pattern by an external magnetic field yields the following version of the central limit theorem.

THEOREM 2.7. *Suppose that $(p^3 \log p)/n \rightarrow 0$. Let $\beta > 0$, $h > 0$, $l \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{N}$. In the case of bounded p , let $|l| \leq p$ and $k \leq p$ in addition. Then, for \mathbb{P}_ξ -almost all ξ , the measures*

$$P_{n, \beta, h e_l, \xi} \left(\sqrt{n} \pi_k \left[\frac{S_n}{n} - x_l^n(\xi) \right] \right)^{-1}$$

converge weakly, as n tends to infinity, to the Gaussian distribution on \mathbb{R}^k with mean zero and covariance matrix

$$C_{\beta, h} = \frac{1 - z(\beta, h)^2}{1 - \beta(1 - z(\beta, h)^2)} \text{Id}_{\mathbb{R}^k}.$$

REMARK 2.8. The central limit theorems formulated in Theorems 2.5 and 2.7 remain valid when the centering $x_l^n(\xi)$ is replaced by

$$\left[\text{Id}_{\mathbb{R}^p} + \frac{1}{1 - \beta(1 - z^2)} \Sigma^n \right] z e_l$$

with $z = z^+(\beta)$ in the case $h = 0$ and $z = z(\beta, h)$, otherwise. This is the linear adjustment of the deterministic centering $z e_l$ which was chosen in [8] in the case of fixed p .

Before turning to the proofs of Theorems 2.5 and 2.7, we will state a result on the asymptotic behavior of the partition function and the probability that the overlap is in a small neighborhood of $z^+(\beta)e_l$ or $z(\beta, h)e_l$, respectively. It is of interest on its own and follows directly from Lemma 3.3.

THEOREM 2.9. (a) *Let $\beta > 0$, $\beta \neq \beta_c$ and $h = 0$. Fix $\varepsilon \in (0, z^+(\beta)/8)$ in the case $\beta > \beta_c$ and $\varepsilon > 0$ arbitrary, otherwise. Then, under the condition $(p^2 \log p)/n \rightarrow 0$, for every $l \in L$ and for \mathbb{P}_ξ -almost all ξ ,*

$$(2.7) \quad \begin{aligned} & P_{n, \beta, \xi} \left(\left\| \frac{S_n}{n} - z^+(\beta)e_l \right\| < \varepsilon \right) \\ &= \frac{1}{Z_{n, \beta, \xi} \sqrt{\det(-\beta D^2 \Phi(\lambda_l^n(\xi)))}} \exp\{n \Phi(\lambda_l^n(\xi))\} [1 + o(1)] \exp\{\mathcal{O}(p \delta_n)\} \end{aligned}$$

and

$$(2.8) \quad Z_{n, \beta, \xi} = \left(\sum_{v \in L} \frac{\exp\{n \Phi(\lambda_v^n(\xi))\}}{\sqrt{\det(-\beta D^2 \Phi(\lambda_v^n(\xi)))}} \right) [1 + o(1)] \exp\{\mathcal{O}(p \delta_n)\}$$

as $n \rightarrow \infty$. The properties of the sequence $\{\delta_n\}_{n \in \mathbb{N}}$ will be specified below.

(b) Let $\beta > 0$ and $h > 0$. Assume that $l \in \mathbb{Z} \setminus \{0\}$ satisfies $|l| \leq p$ in the case of bounded p . Then, under the condition $(p^2 \log p)/n \rightarrow 0$, for every $\varepsilon > 0$ and for \mathbb{P}_ξ -almost all ξ ,

$$(2.9) \quad P_{n, \beta, h e_l, \xi} \left(\left\| \frac{S_n}{n} - z(\beta, h) e_l \right\| < \varepsilon \right) = [1 + o(1)] \exp\{\mathcal{O}(p \delta_n)\}$$

and

$$(2.10) \quad Z_{n, \beta, h e_l, \xi} = \frac{\exp\{n \Phi(\lambda_l^n(\xi))\}}{\sqrt{\det(-\beta D^2 \Phi(\lambda_l^n(\xi)))}} [1 + o(1)] \exp\{\mathcal{O}(p \delta_n)\}$$

as $n \rightarrow \infty$.

The sequence $\{\delta_n\}_{n \in \mathbb{N}}$ in (a) and (b) tends to zero as $n \rightarrow \infty$. In the special case $z = 0$ (i.e., $\beta < \beta_c$ and $h = 0$), $p \delta_n \rightarrow 0$ holds. If the stronger assumption $(p^3 \log p)/n \rightarrow 0$ holds, then $p \delta_n \rightarrow 0$ is always satisfied.

REMARK 2.10. (a) The error terms of the form $o(1)$ in the above statement depend on ξ in the following way: given $\varrho > 0$, there exists for \mathbb{P}_ξ -almost all ξ an $n(\xi, \varrho)$ such that the corresponding error term is bounded by ϱ for all $n \geq n(\xi, \varrho)$. The error terms of the form $\mathcal{O}(p \delta_n)$ depend on ξ in the sense that there exist a constant $C > 0$ and an $n(\xi, C)$ such that the corresponding error term is bounded by $C p \delta_n$ for all $n \geq n(\xi, C)$.

(b) For $\beta < \beta_c$ and $h = 0$ the theorem states that

$$P_{n, \beta, \xi} \left(\left\| \frac{S_n}{n} \right\| < \varepsilon \right) = 1 + o(1)$$

and

$$Z_{n, \beta, \xi} = (1 - \beta)^{-p/2} [1 + o(1)]$$

\mathbb{P}_ξ -almost surely as $n \rightarrow \infty$.

3. Proof of the central limit theorem. We start by proving Proposition 2.1, which controls the influence of the random patterns. The proof relies heavily on the Appendix to [2], where the same problem is considered. In [2], the accent is put on the rate of decay as $n \rightarrow \infty$ of the probability that the norm of the random matrix in question exceeds the given bound $\sqrt{\alpha}(2 + \sqrt{\alpha})(1 + \varepsilon)$, while in Proposition 2.1, we aim for less restrictive conditions on p such that the bound $\sqrt{\alpha}(2 + \sqrt{\alpha})(1 + \varepsilon)$ still holds \mathbb{P}_ξ -almost surely.

PROOF OF PROPOSITION 2.1. First, we consider the case of large p . In [2], Lemma A.1, the following bound on the expectation of the trace of powers of $\Sigma^n(\xi)$ is given: there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$ satisfying $k^6 \leq n$ and $k^4 \leq p$,

$$\mathbb{E}_\xi \text{tr}[\Sigma^n(\xi)]^k \leq C n [\sqrt{\alpha}(2 + \sqrt{\alpha})]^{k+1}.$$

By Chebyshev's inequality, for even k , we have

$$\begin{aligned}\mathbb{P}_\xi(\|\Sigma^n(\xi)\| > \sqrt{\alpha}(2 + \sqrt{\alpha})(1 + \varepsilon)) &\leq [\sqrt{\alpha}(2 + \sqrt{\alpha})(1 + \varepsilon)]^{-k} \mathbb{E}_\xi \operatorname{tr}[\Sigma^n(\xi)]^k \\ &\leq Cn\sqrt{\alpha}(2 + \sqrt{\alpha})(1 + \varepsilon)^{-k}.\end{aligned}$$

For $p \geq (3 \log n / \log(1 + \varepsilon))^4$ and large enough n we may choose an even k satisfying $[3 \log n + \log(3C) + \log \sqrt{\alpha}] / \log(1 + \varepsilon) \leq k \leq p^{1/4}$ because $\log \sqrt{\alpha} \rightarrow -\infty$. Hence, in this case,

$$(3.1) \quad \mathbb{P}_\xi(\|\Sigma^n(\xi)\| > \sqrt{\alpha}(2 + \sqrt{\alpha})(1 + \varepsilon)) \leq \frac{1}{n^2}.$$

To treat the case $p < (3 \log n / \log(1 + \varepsilon))^4$, just note that for fixed n the operator norm of the $(p \times p)$ matrix $\Sigma^n(\xi)$ is a nondecreasing function of the dimension p . Therefore, (2.1) follows directly from (3.1) for all p . Since the right-hand side of (2.1) is summable over n , (2.2) follows by an application of the Borel–Cantelli lemma. \square

Before turning to the proofs of Proposition 2.3, the central limit theorem and Theorem 2.9, we need to investigate the map Φ defined in (2.3). For this purpose, we need the following lemma, which is mainly taken from a preliminary version of [4]. It will serve as an important tool when estimating $(1/n) \sum_{i=1}^n \langle v, \xi_i \rangle^2$ because it gives uniform bounds for the crucial part of the sum when $|\langle v, \xi_i \rangle|$ exceeds a given bound $a > 0$ although $\|v\| \leq \varrho$ is small.

LEMMA 3.1. *There exists a constant $c_3 > 0$ such that for all $\varrho > 0$ and $a > 0$, which both may depend on n , and for \mathbb{P}_ξ -almost all ξ , there exists an $n_2(\xi) \geq n_0(\xi)$ such that for all $n \geq n_2(\xi)$,*

$$\frac{1}{n} \sum_{i=1}^n \langle v, \xi_i \rangle^2 \mathbf{1}_{\{|\langle v, \xi_i \rangle| > a\}} \leq \Gamma(n, a/\varrho) \|v\|^2$$

for all $v \in \mathbb{R}^p$ with $\|v\| \leq \varrho$, where

$$\Gamma(n, a) = 8 \left[\alpha |\log \alpha| + c_3 \alpha + 4 \exp \left\{ -\frac{a^2}{8} \right\} + \frac{2}{n} \log n \right].$$

Since in [4] the proof of the analogous lemma is given only for p increasing at least linearly with $\log n$, we will show that it naturally extends to the case of small p .

PROOF OF LEMMA 3.1. For $v \in \mathbb{R}^p$ set

$$X_a(v) = \frac{1}{n} \sum_{i=1}^n \langle v, \xi_i \rangle^2 \mathbf{1}_{\{|\langle v, \xi_i \rangle| > a\}}.$$

Although we are interested in $X_a(v)$ only, we need to consider the related variables

$$Y_a(v) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{|\langle v, \xi_i \rangle| > a\}}$$

in addition. For fixed $v \neq 0$, we have by Chebyshev's inequality

$$(3.2) \quad \begin{aligned} \mathbb{P}_\xi(X_a(v) \geq x) &\leq \exp\{-txn\}(\mathbb{E}_\xi(\exp\{t\langle v, \xi_1 \rangle^2 1_{\{|\langle v, \xi_1 \rangle| > a\}}\}))^n \\ &\leq \exp\{-txn\}(1 + \mathbb{E}_\xi(\exp\{t\langle v, \xi_1 \rangle^2\} 1_{\{|\langle v, \xi_1 \rangle| > a\}}))^n \end{aligned}$$

for all $x > 0$ and every $t > 0$. Using the estimate $\cosh y \leq \exp\{y^2/2\}$,

$$\mathbb{P}_\xi(\langle v, \xi_1 \rangle > a) \leq \exp\{-sa\} \mathbb{E}_\xi \exp\{s\langle v, \xi_1 \rangle\} \leq \exp\{-sa\} \exp\{s^2 \|v\|^2/2\}$$

follows for every $s > 0$, and therefore, choosing $s = a/\|v\|^2$ yields

$$(3.3) \quad \mathbb{P}_\xi(\langle v, \xi_1 \rangle > a) \leq \exp\{-a^2/(2\|v\|^2)\}.$$

By partial integration,

$$\begin{aligned} &\mathbb{E}_\xi(\exp\{t\langle v, \xi_1 \rangle^2\} 1_{\{|\langle v, \xi_1 \rangle| > a\}}) \\ &= 2 \int_{(a, \infty)} \exp\{tx^2\} \mathbb{P}_\xi(\langle v, \xi_1 \rangle > x) dx \\ &= 2 \exp\{ta^2\} \mathbb{P}_\xi(\langle v, \xi_1 \rangle > a) + 2 \int_{(a, \infty)} 2tx \exp\{tx^2\} \mathbb{P}_\xi(\langle v, \xi_1 \rangle > x) dx. \end{aligned}$$

From (3.3), we find for all $t < (2\|v\|^2)^{-1}$,

$$\begin{aligned} &\mathbb{E}_\xi(\exp\{t\langle v, \xi_1 \rangle^2\} 1_{\{|\langle v, \xi_1 \rangle| > a\}}) \\ &\leq 2 \exp\{ta^2\} \exp\left\{\frac{-a^2}{2\|v\|^2}\right\} + 2 \int_a^\infty 2tx \exp\{tx^2\} \exp\left\{\frac{-x^2}{2\|v\|^2}\right\} dx \\ &= \frac{2}{1 - 2t\|v\|^2} \exp\left\{-a^2\left(\frac{1}{2\|v\|^2} - t\right)\right\}. \end{aligned}$$

Choosing $t = (4\|v\|^2)^{-1}$, from (3.2) it follows that

$$(3.4) \quad \mathbb{P}_\xi(X_a(v) \geq x) \leq \exp\left\{-n \frac{x}{4\|v\|^2}\right\} \left(1 + 4 \exp\left\{-\frac{a^2}{4\|v\|^2}\right\}\right)^n.$$

Another application of Chebyshev's inequality and (3.3) gives

$$\mathbb{P}_\xi(Y_a(v) \geq x) \leq \exp\{-txn\} \left(1 + 2 \exp\{t\} \exp\left\{-\frac{a^2}{2\|v\|^2}\right\}\right)^n$$

for all $x > 0$ and every $t > 0$, and the choice $t = a^2/(4\|v\|^2)$ yields

$$(3.5) \quad \mathbb{P}_\xi(Y_a(v) \geq x) \leq \exp\left\{-n \frac{a^2 x}{4\|v\|^2}\right\} \left(1 + 2 \exp\left\{-\frac{a^2}{4\|v\|^2}\right\}\right)^n.$$

Instead of a fixed v , we finally want to consider v taken from a ball in \mathbb{R}^p , where we are aiming for uniform bounds. As a first step in this direction, we

will consider v from a finite set. Denote by $\mathscr{W}_{p,r}$ the cubic lattice in \mathbb{R}^p with spacing r/\sqrt{p} , and for $r < r_0$ define $\mathscr{W}_{p,r}(r_0) = \mathscr{W}_{p,r} \cap \overline{B_{r_0}(0)}$. Then there exists a constant $c > 0$ such that

$$(3.6) \quad |\mathscr{W}_{p,r}(r_0)| \leq \exp\{p(\log(r_0/r) + c)\}.$$

This estimate allows us to estimate $X_a(v)$ and $Y_a(v)$ uniformly on the grid $\mathscr{W}_{p,r}(r_0)$. By (3.4),

$$(3.7) \quad \begin{aligned} \mathbb{P}_\xi \left(\sup_{v \in \mathscr{W}_{p,r}(r_0)} X_a(v) \geq 4r_0^2 \left[\alpha \left(\log \frac{r_0}{r} + c \right) + 4 \exp \left\{ -\frac{a^2}{4r_0^2} \right\} + \frac{2}{n} \log n \right] \right) \\ \leq \frac{1}{n^2}, \end{aligned}$$

and, similarly for each $k \in \mathbb{N}$ by (3.5),

$$(3.8) \quad \mathbb{P}_\xi \left(\sup_{v \in \mathscr{W}_{p,r}(r_0)} Y_a(v) \geq 4 \frac{r_0^2}{a^2} \left[\alpha \left(\log \frac{r_0}{r} + c \right) + 2 + \frac{2}{n} \log(nk) \right] \right) \leq \frac{1}{(nk)^2}.$$

Instead of taking the supremum over the grid $\mathscr{W}_{p,r}(r_0)$, we want to consider the supremum over all $v \in \overline{B_{r_0}(0)}$. For this purpose, we will introduce a sequence of lattices \mathscr{W}_{p,r_k} , $k \in \mathbb{N}_0$, with r_0 and $r_k \searrow 0$ specified below. This allows us to rewrite each $v \in \overline{B_{r_0}(0)}$ for arbitrary $k_0 \in \mathbb{N}$ as

$$(3.9) \quad v = \sum_{k=1}^{k_0} v_k + v_{k_0+1}$$

with $v_k \in \mathscr{W}_{p,r_k}(r_{k-1})$ for $1 \leq k \leq k_0$ and $\|v_{k_0+1}\| \leq r_{k_0}$. Such a representation will be used together with the following estimates on $X_a(v)$ and $Y_a(v)$. For $a_1, \tilde{a} > 0$ such that $a = a_1 + \tilde{a}$ and for arbitrary vectors $v, v_1, \tilde{v} \in \mathbb{R}^p$ with $v = v_1 + \tilde{v}$ we have

$$(3.10) \quad \begin{aligned} X_a(v) &\leq 2X_{a_1}(v_1) + \frac{2}{n} \sum_{i=1}^n \langle v_1, \xi_i \rangle^2 \mathbf{1}_{\{|\langle v, \xi_i \rangle| > a\}} \mathbf{1}_{\{|\langle v_1, \xi_i \rangle| \leq a_1\}} + \frac{2}{n} \sum_{i=1}^n \langle \tilde{v}, \xi_i \rangle^2 \\ &\leq 2X_{a_1}(v_1) + 2a_1^2 Y_{\tilde{a}}(\tilde{v}) + 2 \left\langle \tilde{v}, \frac{1}{n} \sum_{i=1}^n \xi_i \xi_i^T \tilde{v} \right\rangle \end{aligned}$$

and

$$(3.11) \quad Y_a(v) \leq Y_{a_1}(v_1) + Y_{\tilde{a}}(\tilde{v}).$$

Note that the bound (3.10) on X_a is weaker than the one given in [4]. This weaker bound will suffice for our purpose.

Next, we choose $r_k = \alpha^k$ for defining the lattice \mathscr{W}_{p,r_k} , $k \in \mathbb{N}_0$, and $a_1 = a/\sqrt{2}$. Set $\gamma = 1 - 1/\sqrt{2}$ as an abbreviation. By (3.10) with $v_1 \in \mathscr{W}_{p,r_1}(r_0)$ and the estimates (3.7) and (2.1), we find for arbitrary $y > 0$ and all $n \geq \tilde{n}_0$ with

\tilde{n}_0 from Proposition 2.1,

$$\begin{aligned}
 (3.12) \quad & \mathbb{P}_\xi \left(\sup_{\|v\| \leq 1} X_a(v) \geq 8 \left[\alpha(|\log r_1| + c) + 4 \exp \left\{ -\frac{a_1^2}{4} \right\} + \frac{2}{n} \log n \right] \right. \\
 & \quad \left. + 2a_1^2 y + 2r_1^2(1 + \varepsilon_n) \right) \\
 & \leq \frac{1}{n^2} + \mathbb{P}_\xi \left(\sup_{\|\tilde{v}\| \leq r_1} Y_{\gamma a}(\tilde{v}) \geq y \right) + \mathbb{P}_\xi \left(\left\| \frac{1}{n} \sum_{i=1}^n \xi_i \xi_i^T \right\| \geq 1 + \varepsilon_n \right) \\
 & \leq \frac{2}{n^2} + \mathbb{P}_\xi \left(\sup_{\|\tilde{v}\| \leq r_1} Y_{\gamma a}(\tilde{v}) \geq y \right).
 \end{aligned}$$

It remains to choose an appropriate y and to estimate the last probability in (3.12) afterward. For this purpose, we choose a decreasing sequence $(d_k)_{k \in \mathbb{N}}$ by setting $d_k = \gamma a 2^{-(k-1)}$ for $k \in \mathbb{N}$. Let n be large enough to satisfy $2\alpha < 1$. Then there exists a $k_0 = k_0(n) \geq 2$ such that $d_{k_0} \geq \sqrt{p} r_{k_0}$ and, therefore $Y_{d_{k_0}}(w) = 0$ for all $\|w\| \leq r_{k_0}$. By the estimate (3.11) and the representation (3.9),

$$(3.13) \quad \sup_{\|\tilde{v}\| \leq r_1} Y_{\gamma a}(\tilde{v}) \leq \sum_{k=2}^{k_0} \sup_{v_k \in \mathcal{W}_{p, r_k}(r_{k-1})} Y_{d_k}(v_k)$$

follows. Since $\alpha \leq 1/3$ for large enough n ,

$$\sum_{k=2}^{\infty} 4 \left(\frac{r_{k-1}}{d_k} \right)^2 \left[\alpha \left(\log \frac{r_{k-1}}{r_k} + c \right) + 2 + \frac{2}{n} \log(nk) \right] \leq \tilde{c}_3 \frac{\alpha^2}{a^2}$$

holds for these n with some constant $\tilde{c}_3 > 0$. From (3.8) and (3.13) it follows that

$$\mathbb{P}_\xi \left(\sup_{\|\tilde{v}\| \leq r_1} Y_{\gamma a}(\tilde{v}) \geq \tilde{c}_3 \frac{\alpha^2}{a^2} \right) \leq \sum_{k=2}^{\infty} \frac{1}{(nk)^2} \leq \frac{1}{n^2} \frac{\pi^2}{6}$$

and, therefore, returning to (3.12), we find for all $n \geq \tilde{n}_0$ with $\alpha \leq 1/3$,

$$\mathbb{P}_\xi \left(\sup_{\|v\| \leq 1} X_a(v) \geq \Gamma(n, a) \right) \leq \left(2 + \frac{\pi^2}{6} \right) \frac{1}{n^2}.$$

An application of the Borel–Cantelli lemma shows the existence of an $n_2(\xi)$ such that

$$(3.14) \quad \sup_{\|v\| \leq 1} X_a(v) \leq \Gamma(n, a)$$

for \mathbb{P}_ξ -almost all ξ and all $n \geq n_2(\xi)$. Applying (3.14) with v replaced by the normalized vector $v/\|v\|$ for $v \in \overline{B_\varrho(0)} \setminus \{0\}$ yields

$$X_a(v) \leq \sup_{\|\tilde{v}\| \leq 1} X_{a/\varrho}(\tilde{v}) \|v\|^2 \leq \Gamma(n, a/\varrho) \|v\|^2$$

for all $v \in \mathbb{R}^p$ with $\|v\| \leq \varrho$. \square

As a first application of the preceding lemma, we can now prove the existence of a unique maximum of Φ in a small neighborhood of $\operatorname{arctanh} z e_l$, where z is defined as in Proposition 2.3. The case $h = 0$ and p increasing at least linearly with $\log n$ has been proved in [4] by showing local convexity of $-\Phi$. The proof naturally extends to the case $h \neq 0$.

PROOF OF PROPOSITION 2.3. First note that

$$(3.15) \quad D^2\Phi(\lambda) = -\frac{1}{\beta} \operatorname{Id}_{\mathbb{R}^p} + \frac{1}{n} \sum_{i=1}^n \cosh^{-2}(\langle \lambda, \xi_i \rangle) \xi_i \xi_i^T.$$

To prove that $-D^2\Phi(\lambda)$ is positive definite for λ in a neighborhood of $\operatorname{arctanh} z e_l$, we choose $u, v \in \mathbb{R}^p \setminus \{0\}$ arbitrarily. Then

$$\begin{aligned} & \langle u, -D^2\Phi(\operatorname{arctanh} z e_l + v)u \rangle \\ &= \frac{1}{\beta} \|u\|^2 - \frac{1}{n} \sum_{i=1}^n (1 - z^2) \langle u, \xi_i \xi_i^T u \rangle \\ & \quad - \frac{1}{n} \sum_{i=1}^n [\cosh^{-2}(\operatorname{arctanh} z \langle e_l, \xi_i \rangle + \langle v, \xi_i \rangle) \\ & \quad \quad - \cosh^{-2}(\operatorname{arctanh} z \langle e_l, \xi_i \rangle)] \langle u, \xi_i \rangle^2, \end{aligned}$$

where we used $\cosh^{-2} \operatorname{arctanh} z = 1 - z^2$. Applying the estimate

$$(3.16) \quad |\cosh^{-2}(x + y) - \cosh^{-2}(x)| \leq \min\{1, 2|y|\},$$

the bound from Proposition 2.1 on the random matrix and Lemma 3.1, we find that for every auxiliary $a > 0$, \mathbb{P}_ξ -almost all ξ and all $n \geq n_2(\xi)$,

$$\begin{aligned} & \langle u, -D^2\Phi(\operatorname{arctanh} z e_l + v)u \rangle \\ & \geq \left[\frac{1}{\beta} (1 - \beta(1 - z^2)(1 + \varepsilon_n)) \right] \|u\|^2 \\ & \quad - \frac{2}{n} \sum_{i=1}^n |\langle v, \xi_i \rangle| \langle u, \xi_i \rangle^2 \mathbf{1}_{\{|\langle v, \xi_i \rangle| \leq a\|v\|\}} - \frac{1}{n} \sum_{i=1}^n \langle u, \xi_i \rangle^2 \mathbf{1}_{\{|\langle v, \xi_i \rangle| > a\|v\|\}} \\ & \geq \left[\frac{1}{\beta} (1 - \beta(1 - z^2)(1 + \varepsilon_n)) - 2a\|v\|(1 + \varepsilon_n) \right] \|u\|^2 \\ & \quad - \frac{1}{n} \sum_{i=1}^n \max \left\{ \left\langle \frac{u}{\|u\|}, \xi_i \right\rangle^2 \mathbf{1}_{\{|\langle u, \xi_i \rangle| > a\|u\|\}}, \left\langle \frac{v}{\|v\|}, \xi_i \right\rangle^2 \mathbf{1}_{\{|\langle v, \xi_i \rangle| > a\|v\|\}} \right\} \|u\|^2 \\ & \geq \left[\frac{1}{\beta} (1 - \beta(1 - z^2)(1 + \varepsilon_n)) - 2a\|v\|(1 + \varepsilon_n) - 2\Gamma(n, a) \right] \|u\|^2. \end{aligned}$$

Choosing n and a large makes $\Gamma(n, a)$ arbitrarily small and ε_n tends to zero as $n \rightarrow \infty$. Therefore, there exist $a > 0$ and $\tilde{n}_1 \in \mathbb{N}$ such that

$$(1 - z^2)\varepsilon_n + 2\Gamma(n, a) \leq \frac{1}{2} \left[\frac{1}{\beta}(1 - \beta(1 - z^2)) - c_1 \right]$$

for all $n \geq \tilde{n}_1$. Having fixed a , there exists an r_1 such that

$$2a\|v\|(1 + \varepsilon_n) \leq \frac{1}{2} \left[\frac{1}{\beta}(1 - \beta(1 - z^2)) - c_1 \right]$$

holds for all $n \geq \tilde{n}_1$ and all v satisfying $\|v\| \leq r_1$. This proves

$$(3.17) \quad \langle u, -D^2\Phi(\operatorname{arctanh} z e_l + v)u \rangle \geq c_1\|u\|^2$$

for \mathbb{P}_ξ -almost all ξ , all $u \in \mathbb{R}^p$, $\|v\| \leq r_1$ and all $n \geq n_1(\xi) = \max\{n_2(\xi), \tilde{n}_1\}$.

Next, we will show that Φ does not attain its maximum on the boundary of $B_{r_1}(\operatorname{arctanh} z e_l)$. By Taylor's formula, for all $v \in B_{r_1}(\operatorname{arctanh} z e_l)$, there exists a $\theta \in [0, 1]$ such that

$$\begin{aligned} & \Phi(\operatorname{arctanh} z e_l + v) \\ &= \Phi(\operatorname{arctanh} z e_l) + \langle \Sigma^n(\xi) z e_l, v \rangle - \frac{1}{2} \langle v, -D^2\Phi(\operatorname{arctanh} z e_l + \theta v)v \rangle \\ &\leq \Phi(\operatorname{arctanh} z e_l) + |z|\varepsilon_n\|v\| - \frac{c_1}{2}\|v\|^2 \end{aligned}$$

for all $n \geq n_1(\xi)$ by our bound on $\Sigma^n(\xi)$ and the estimate (3.17). Therefore,

$$\Phi(\operatorname{arctanh} z e_l + v) < \Phi(\operatorname{arctanh} z e_l)$$

for all v satisfying $c_2\varepsilon_n < \|v\| \leq r_1$ with $c_2 = 2|z|/c_1$. This establishes the existence of a unique maximum of Φ on the set $\overline{B_{r_1}(\operatorname{arctanh} z e_l)}$ as well as (2.4). \square

As an abbreviation, for every $\nu \in L$, set

$$(3.18) \quad C_\nu^n(\xi) = -D^2\Phi(\lambda_\nu^n(\xi)) = \frac{1}{\beta} \operatorname{Id}_{\mathbb{R}^p} - \frac{1}{n} \sum_{i=1}^n \cosh^{-2}(\langle \lambda_\nu^n(\xi), \xi_i \rangle) \xi_i \xi_i^T.$$

The following lemma shows that $C_\nu^n(\xi)$ converges to a deterministic diagonal matrix provided that $\sqrt{p}\varepsilon_n \rightarrow 0$.

LEMMA 3.2. *Let $\beta > 0$ and $h \in \mathbb{R}$ such that $(\beta, h) \neq (\beta_c, 0)$. Choose an $l \in \mathbb{Z} \setminus \{0\}$, satisfying $|l| \leq p$ in the case of bounded p . Then there exists a constant $c_4 > 0$ such that*

$$\sup_{\nu \in L} \left\| C_\nu^n(\xi) - \frac{1}{\beta} [1 - \beta(1 - z^2)] \operatorname{Id}_{\mathbb{R}^p} \right\| \leq c_4 \sqrt{p} \varepsilon_n$$

for \mathbb{P}_ξ -almost all ξ and all $n \geq n_1(\xi)$.

PROOF. By the identity $\cosh^{-2} \operatorname{arctanh} z = 1 - z^2$, (3.16), (2.4) and Proposition 2.1, for \mathbb{P}_ξ -almost all ξ , all $n \geq n_1(\xi)$ and all $\nu \in L$,

$$\begin{aligned} & \left\| C_\nu^n(\xi) - \frac{1}{\beta} [1 - \beta(1 - z^2)] \operatorname{Id}_{\mathbb{R}^p} \right\| \\ & \leq \left\| \frac{1}{n} \sum_{i=1}^n [\cosh^{-2}(\langle \lambda_\nu^n(\xi), \xi_i \rangle) - \cosh^{-2}(\langle \operatorname{arctanh} z e_\nu, \xi_i \rangle)] \xi_i \xi_i^T \right\| \\ & \quad + (1 - z^2) \left\| \frac{1}{n} \sum_{i=1}^n \xi_i \xi_i^T - \operatorname{Id}_{\mathbb{R}^p} \right\| \\ & \leq 2 \|\lambda_\nu^n(\xi) - \operatorname{arctanh} z e_\nu\| \sqrt{p} \left\| \frac{1}{n} \sum_{i=1}^n \xi_i \xi_i^T \right\| + \|\Sigma^n\| \\ & \leq [2c_2 \sqrt{p}(1 + \varepsilon_n) + 1] \varepsilon_n. \end{aligned} \quad \square$$

The proof of the central limit theorem as well as the proof of Theorem 2.9 is based on the following lemma, which describes the asymptotic behavior of certain integrals. We will give the proof in Section 4.

LEMMA 3.3. *Let $\beta > 0$ and $h \in \mathbb{R}$ be such that $(\beta, h) \neq (\beta_c, 0)$, and let $l \in \mathbb{Z} \setminus \{0\}$, $y \in \mathbb{R}^k$ and $k \geq |l|$. Assume $k \leq p$ in the case of bounded p . As an abbreviation, set*

$$y_n = \frac{1}{\sqrt{n}} \begin{pmatrix} y \\ 0 \end{pmatrix} \in \mathbb{R}^p.$$

For all $\lambda \in \mathbb{R}^p$, define the ξ -dependent function Ψ by

$$(3.19) \quad \Psi(\lambda) = -\frac{1}{2\beta} \|\lambda - h e_l - y_n\|^2 + \frac{1}{n} \sum_{i=1}^n \log \cosh \langle \lambda, \xi_i \rangle.$$

Then there exist $R_0, r_0 > 0$, depending on β, h and y only, such that for all $R \geq R_0$, all $r \leq r_0$ and \mathbb{P}_ξ -almost all ξ the following assertions hold:

(a) Under the general assumption $p/n \rightarrow 0$,

$$\int_{\bigcap_{\nu \in L} B_r^c(\operatorname{arctanh} z e_\nu)} \exp\{n\Psi(\lambda)\} d\lambda = o(1) \int_{\mathbb{R}^p} \exp\{n\Psi(\lambda)\} d\lambda$$

as $n \rightarrow \infty$.

(b) Again under the general assumption $p/n \rightarrow 0$, for every $\nu \in L$,

$$\int_{B_r(\operatorname{arctanh} z e_\nu)} \exp\{n\Psi(\lambda)\} d\lambda = [1 + o(1)] \int_{B_{Re_n}(\operatorname{arctanh} z e_\nu)} \exp\{n\Psi(\lambda)\} d\lambda$$

and

$$\begin{aligned} & \int_{\bigcup_{\nu \in L} B_r(\operatorname{arctanh} z e_\nu)} \exp\{n\Psi(\lambda)\} d\lambda \\ & = [1 + o(1)] \int_{\bigcup_{\nu \in L} B_{Re_n}(\operatorname{arctanh} z e_\nu)} \exp\{n\Psi(\lambda)\} d\lambda \end{aligned}$$

as $n \rightarrow \infty$.

(c) Furthermore, under the condition $(p^2 \log p)/n \rightarrow 0$, for every $v \in L$,

$$\begin{aligned} & \left(\frac{n}{2\pi\beta} \right)^{p/2} \int_{B_{Re_n}(\operatorname{arctanh} z e_v)} \exp\{n\Psi(\lambda)\} d\lambda \\ &= \det(\beta C_v^n(\xi))^{-1/2} \exp\{n[\Phi(\lambda_v^n(\xi)) + \langle x_v^n(\xi), y_n \rangle]\} \\ & \quad \times \exp\left\{-\frac{n}{2\beta} \left\langle y_n, \left[\operatorname{Id}_{\mathbb{R}^p} - \frac{1}{\beta} C_v^n(\xi)^{-1} \right] y_n \right\rangle\right\} \\ & \quad \times [1 + o(1)] \exp\{\mathcal{O}(p\delta_n)\} \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{n}{2\pi\beta} \right)^{p/2} \int_{\mathbb{R}^p} \exp\{n\Psi(\lambda)\} d\lambda \\ &= \left(\sum_{v \in L} \det(\beta C_v^n(\xi))^{-1/2} \exp\{n[\Phi(\lambda_v^n(\xi)) + \langle x_v^n(\xi), y_n \rangle]\} \right. \\ & \quad \times \exp\left\{-\frac{n}{2\beta} \left\langle y_n, \left[\operatorname{Id}_{\mathbb{R}^p} - \frac{1}{\beta} C_v^n(\xi)^{-1} \right] y_n \right\rangle\right\} \Bigg) \\ & \quad \times [1 + o(1)] \exp\{\mathcal{O}(p\delta_n)\} \end{aligned}$$

as $n \rightarrow \infty$ with a sequence $\{\delta_n\}_{n \in \mathbb{N}}$ tending to zero, which has the same properties as the one in Theorem 2.9.

The following lemma will be used to replace the indicator function of a ball by its smoothed version obtained via the convolution with the continuous density of a Gaussian random variable.

LEMMA 3.4. Let $\beta > 0$, $n, p \in \mathbb{N}$ and $\varepsilon > 0$. Then for all $x \in \mathbb{R}^p$,

$$\begin{aligned} & \left(\frac{n\beta}{2\pi} \right)^{p/2} \int_{\mathbb{R}^p} 1_{B_{\varepsilon/2}(0)}(x + \lambda) \exp\left\{-\frac{n\beta}{2} \|\lambda\|^2\right\} d\lambda - 2^{p/2} \exp\left\{-\frac{n\beta}{16} \varepsilon^2\right\} \\ & \leq 1_{B_\varepsilon(0)}(x) \\ & \leq \left(\frac{n\beta}{2\pi} \right)^{p/2} \int_{\mathbb{R}^p} 1_{B_{2\varepsilon}(0)}(x + \lambda) \exp\left\{-\frac{n\beta}{2} \|\lambda\|^2\right\} d\lambda + 2^{p/2} \exp\left\{-\frac{n\beta}{4} \varepsilon^2\right\}. \end{aligned}$$

PROOF. By splitting the domain of integration into the two sets with $\|\lambda\| < \varepsilon/2$ and $\|\lambda\| \geq \varepsilon/2$, we find

$$\begin{aligned}
 & \left(\frac{n\beta}{2\pi}\right)^{p/2} \int_{\mathbb{R}^p} 1_{B_{\varepsilon/2}(0)}(x + \lambda) \exp\left\{-\frac{n\beta}{2}\|\lambda\|^2\right\} d\lambda \\
 & \leq \left(\frac{n\beta}{2\pi}\right)^{p/2} \left[1_{B_\varepsilon(0)}(x) \int_{\mathbb{R}^p} \exp\left\{-\frac{n\beta}{2}\|\lambda\|^2\right\} d\lambda \right. \\
 & \quad \left. + \int_{\mathbb{R}^p} 1_{B_{\varepsilon/2}^c(0)}(\lambda) \exp\left\{-\frac{n\beta}{2}\|\lambda\|^2\right\} d\lambda \right] \\
 & \leq 1_{B_\varepsilon(0)}(x) + \exp\left\{-\frac{n\beta}{4}\left(\frac{\varepsilon}{2}\right)^2\right\} \left(\frac{n\beta}{2\pi}\right)^{p/2} \int_{\mathbb{R}^p} \exp\left\{-\frac{n\beta}{4}\|\lambda\|^2\right\} d\lambda \\
 & = 1_{B_\varepsilon(0)}(x) + 2^{p/2} \exp\left\{-\frac{n\beta}{4}\left(\frac{\varepsilon}{2}\right)^2\right\}.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & \left(\frac{n\beta}{2\pi}\right)^{p/2} \int_{\mathbb{R}^p} 1_{B_{2\varepsilon}(0)}(x + \lambda) \exp\left\{-\frac{n\beta}{2}\|\lambda\|^2\right\} d\lambda \\
 & \geq \left(\frac{n\beta}{2\pi}\right)^{p/2} 1_{B_\varepsilon(0)}(x) \int_{\mathbb{R}^p} 1_{B_\varepsilon(0)}(\lambda) \exp\left\{-\frac{n\beta}{2}\|\lambda\|^2\right\} d\lambda \\
 & = 1_{B_\varepsilon(0)}(x) \left[1 - \left(\frac{n\beta}{2\pi}\right)^{p/2} \int_{\mathbb{R}^p} 1_{B_\varepsilon^c(0)}(\lambda) \exp\left\{-\frac{n\beta}{2}\|\lambda\|^2\right\} d\lambda \right] \\
 & \geq 1_{B_\varepsilon(0)}(x) - 2^{p/2} \exp\left\{-\frac{n\beta}{4}\varepsilon^2\right\}. \quad \square
 \end{aligned}$$

Now, we are ready to give the proof of the central limit theorem. Let us start with the case of the unbiased Hamiltonian.

PROOF OF THEOREM 2.5. First note that we may restrict ourselves to the case $k \geq |l|$. Furthermore, we may assume that $p \geq k$. By [12], Lemma C1, it suffices to show for \mathbb{P}_ξ -almost all ξ the pointwise convergence of the Laplace transforms of

$$Q_n = P_{n, \beta, \xi} \left(\sqrt{n} \pi_k \left[\frac{S_n}{n} - x_l^n(\xi) \right] \in \cdot \mid \left\| \frac{S_n}{n} - z e_l \right\| < \varepsilon \right)$$

[with $z = z^+(\beta)$] to the Laplace transform of the Gaussian measure on \mathbb{R}^k with mean zero and covariance matrix C_β . For this purpose, let $y \in \mathbb{R}^k$ and define $y_n = (1/\sqrt{n}) \binom{y}{0} \in \mathbb{R}^p$ as in Lemma 3.3. Then the Laplace transform $Q_n^*(y)$ of Q_n at y equals

$$Q_n^*(y) = \zeta_{n, \varepsilon} \int \exp \left\{ n \left\langle \frac{S_n}{n} - x_l^n(\xi), y_n \right\rangle \right\} 1_{\{\|(S_n/n) - z e_l\| < \varepsilon\}} dP_{n, \beta, \xi},$$

where

$$\zeta_{n,\varepsilon} = \left[P_{n,\beta,\xi} \left(\left\| \frac{S_n}{n} - ze_l \right\| < \varepsilon \right) \right]^{-1}.$$

With the help of Lemma 3.4, the Laplace transform $Q_n^*(y)$ can be estimated from above, respectively, below by

$$\begin{aligned} & \frac{1}{Z_{n,\beta,\xi}} \zeta_{n,\varepsilon} \left(\frac{n\beta}{2\pi} \right)^{p/2} \\ & \times \int \int_{\mathbb{R}^p} 1_{B_{2\pm 1\varepsilon}(0)} \left(\frac{S_n}{n} - ze_l + \lambda \right) \\ & \times \exp \left\{ n \left[-\frac{\beta}{2} \|\lambda\|^2 + \frac{\beta}{2} \left\| \frac{S_n}{n} \right\|^2 + \left\langle \frac{S_n}{n} - x_l^n(\xi), y_n \right\rangle \right] \right\} d\lambda d\mathbb{P}_\sigma \\ & \pm \zeta_{n,\varepsilon} 2^{p/2} \exp \left\{ -\frac{n\beta}{16} \varepsilon^2 + 2\sqrt{nk} \|y\| \right\}, \end{aligned}$$

where we applied

$$\left| \left\langle \frac{S_n}{n} - x_l^n(\xi), y_n \right\rangle \right| \leq 2\sqrt{\frac{k}{n}} \|y\|.$$

Substituting $(1/\beta)(\lambda - y_n) - (1/n)S_n$ for λ and using $p/n \rightarrow 0$, we see that $Q_n^*(y)$ is bounded by

$$\begin{aligned} & \frac{1}{Z_{n,\beta,\xi}} \zeta_{n,\varepsilon} \left(\frac{n}{2\pi\beta} \right)^{p/2} \exp \{ -n \langle x_l^n(\xi), y_n \rangle \} \\ & \times \int_{\{\|\lambda - \beta ze_l\| < 2\pm 1\beta\varepsilon \pm \|y_n\|\}} \exp \left\{ n \left[-\frac{1}{2\beta} \|\lambda - y_n\|^2 + \left\langle \frac{S_n}{n}, \lambda \right\rangle \right] \right\} d\mathbb{P}_\sigma d\lambda \\ & \pm \zeta_{n,\varepsilon} \exp \{ -c_5 n \} \end{aligned}$$

with some constant $c_5 > 0$. Let n be large enough to satisfy $\|y_n\| \leq \beta\varepsilon/4$. Performing the integration with respect to \mathbb{P}_σ (cf. [8], for instance) and using the Curie–Weiss equation $\beta z = \operatorname{arctanh} z$ shows that $Q_n^*(y)$ is bounded by

$$\begin{aligned} & \frac{1}{Z_{n,\beta,\xi}} \zeta_{n,\varepsilon} \left(\frac{n}{2\pi\beta} \right)^{p/2} \exp \{ -n \langle x_l^n(\xi), y_n \rangle \} \\ & \times \int_{\{\|\lambda - \operatorname{arctanh} ze_l\| < 2\pm 2\beta\varepsilon\}} \exp \left\{ n \left[-\frac{1}{2\beta} \|\lambda - y_n\|^2 + \frac{1}{n} \sum_{i=1}^n \log \cosh \langle \lambda, \xi_i \rangle \right] \right\} d\lambda \\ & \pm \zeta_{n,\varepsilon} \exp \{ -c_5 n \}. \end{aligned}$$

Applying Lemma 3.3(b), (c) and Theorem 2.9(a) yields

$$\begin{aligned} (3.20) \quad Q_n^*(y) &= \exp \left\{ -\frac{n}{2\beta} \left\langle y_n, \left[\operatorname{Id}_{\mathbb{R}^p} - \frac{1}{\beta} C_l^n(\xi)^{-1} \right] y_n \right\rangle \right\} [1 + o(1)] \\ &+ \mathcal{O}(\zeta_{n,\varepsilon} \exp \{ -c_5 n \}). \end{aligned}$$

It remains to investigate the error term. By Theorem 2.9,

$$\zeta_{n,\varepsilon} = \sum_{\nu \in L} \left(\frac{\det(\beta C_\nu^n(\xi))}{\det(\beta C_l^n(\xi))} \right)^{-1/2} \exp\{n[\Phi(\lambda_\nu^n(\xi)) - \Phi(\lambda_l^n(\xi))]\}[1 + o(1)].$$

Lemma 3.2 implies

$$\left(\frac{\det(\beta C_\nu^n(\xi))}{\det(\beta C_l^n(\xi))} \right)^{-1/2} \leq (1 + \mathcal{O}(\sqrt{p}\varepsilon_n))^p,$$

where the error term $\mathcal{O}(\sqrt{p}\varepsilon_n)$ does not depend on $\nu \in L$. Furthermore, we know that $\Phi(\operatorname{arctanh} z e_\nu) = f_{\text{CW}}(\beta)$ does not depend on ν . By the definition of $\lambda_\nu^n(\xi)$ as a local maximum of Φ [see (2.4), (3.15) and Proposition 2.1, we find for \mathbb{P}_ξ -almost all ξ ,

$$\begin{aligned} & |\Phi(\lambda_\nu^n(\xi)) - \Phi(\lambda_l^n(\xi))| \\ & \leq 2 \max_{\mu \in L} |\Phi(\lambda_\mu^n(\xi)) - \Phi(\operatorname{arctanh} z e_\mu)| \\ & \leq \max_{\mu \in L} \max\{\|D^2\Phi(\lambda)\| \|\lambda_\mu^n(\xi) - \operatorname{arctanh} z e_\mu\|^2 \mid \lambda \in B_{c_2\varepsilon_n}(\operatorname{arctanh} z e_\mu)\} \\ & \leq \left(\frac{1}{\beta} + 1 + \varepsilon_n\right)(c_2\varepsilon_n)^2 = \mathcal{O}(\varepsilon_n^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \zeta_{n,\varepsilon} & \leq 2p(1 + \mathcal{O}(\sqrt{p}\varepsilon_n))^p \exp\{n\mathcal{O}(\varepsilon_n^2)\} \\ & = 2 \exp\{\log p + o(p) + o(n)\} = 2 \exp\{o(n)\}, \end{aligned}$$

which implies, by (3.20) and Lemma 3.2,

$$\begin{aligned} Q_n^*(y) & = \exp\left\{-\frac{n}{2\beta}\left\langle y_n, \left[\operatorname{Id}_{\mathbb{R}^p} - \frac{1}{\beta}C_l^n(\xi)^{-1}\right]y_n \right\rangle\right\}[1 + o(1)] + o(1) \\ & \rightarrow \exp\left\{\frac{1}{2}\langle y, C_\beta y \rangle\right\} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since $y \mapsto \exp\{\frac{1}{2}\langle y, C_\beta y \rangle\}$ is the Laplace transform of the Gaussian distribution with mean zero and covariance matrix C_β , the proof is complete. \square

Next we give the proof of the central limit theorem in the case of an external magnetic field.

PROOF OF THEOREM 2.7. As in the preceding proof, we will show for \mathbb{P}_ξ -almost all ξ the pointwise convergence of the Laplace transforms Q_n^* of the measures

$$Q_n = P_{n,\beta,he_l,\xi} \left(\sqrt{n}\pi_k \left[\frac{S_n}{n} - x_l^n(\xi) \right] \right)^{-1}.$$

As before, we may assume $k \geq |l|$. Let again $y \in \mathbb{R}^k$ and $y_n = (1/\sqrt{n})\binom{y}{0} \in \mathbb{R}^p$. Then the Laplace transform $Q_n^*(y)$ of Q_n at y equals

$$Q_n^*(y) = \frac{1}{Z_{n,\beta,he_l,\xi}} \int \exp \left\{ n \left\langle \frac{S_n}{n} - x_l^n(\xi), y_n \right\rangle \right\} \\ \times \exp \left\{ \frac{n\beta}{2} \left\| \frac{S_n}{n} \right\|^2 + n \left\langle \frac{S_n}{n}, he_l \right\rangle \right\} d\mathbb{P}_\sigma,$$

and by a Gaussian integration,

$$Q_n^*(y) = \frac{1}{Z_{n,\beta,he_l,\xi}} \left(\frac{n}{2\pi\beta} \right)^{p/2} \exp \{ -n \langle x_l^n(\xi), y_n \rangle \} \\ \times \int_{\mathbb{R}^p} \int \exp \left\{ n \left[-\frac{1}{2\beta} \|\lambda - he_l - y_n\|^2 + \left\langle \frac{S_n}{n}, \lambda \right\rangle \right] \right\} d\mathbb{P}_\sigma d\lambda.$$

Performing the integration with respect to \mathbb{P}_σ as before,

$$Q_n^*(y) = \frac{1}{Z_{n,\beta,he_l,\xi}} \left(\frac{n}{2\pi\beta} \right)^{p/2} \exp \{ -n \langle x_l^n(\xi), y_n \rangle \} \\ \times \int_{\mathbb{R}^p} \exp \left\{ n \left[-\frac{1}{2\beta} \|\lambda - he_l - y_n\|^2 + \frac{1}{n} \sum_{i=1}^n \log \cosh \langle \lambda, \xi_i \rangle \right] \right\} d\lambda.$$

Applying Lemma 3.3(c) and Theorem 2.9(b), we find

$$Q_n^*(y) = \exp \left\{ -\frac{n}{2\beta} \left\langle y_n, \left[\text{Id}_{\mathbb{R}^p} - \frac{1}{\beta} C_\nu^n(\xi)^{-1} \right] y_n \right\rangle \right\} [1 + o(1)]$$

and, therefore,

$$Q_n^*(y) \rightarrow \exp \left\{ \frac{1}{2} \langle y, C_{\beta,h} y \rangle \right\} \quad \text{as } n \rightarrow \infty$$

by Lemma 3.2, which proves the central limit theorem. \square

PROOF OF THEOREM 2.9. The proof of (2.7) and (2.9) follows from Lemma 3.3 (with $y = 0$) with the help of Lemma 3.4 in the same way as the central limit theorem in the case of the unbiased Hamiltonian is proved, and the identities (2.8) and (2.10) can be proved via Lemma 3.3 in the same way as the central limit theorem in the case of an external magnetic field. \square

4. Proof of Lemma 3.3. The whole section is devoted to the proof of Lemma 3.3. Recall that we want to investigate the asymptotic behavior of integrals of the type

$$(4.1) \quad \left(\frac{n}{2\pi\beta} \right)^{p/2} \int_{\mathbb{R}^p} \exp \{ n \Psi(\lambda) \} d\lambda,$$

where Ψ is given by (3.19). Considering $y_n = (1/\sqrt{n})\binom{y}{0}$ as an error term, Proposition 2.3 suggests the following way of splitting the domain of integration into three parts: the outer region

$$U_r = \left(\bigcup_{\nu \in L} B_r(\operatorname{arctanh} z e_\nu) \right)^c,$$

the intermediate region

$$V_{r,R} = U_r^c \setminus \bigcup_{\nu \in L} B_{R\varepsilon_n}(\operatorname{arctanh} z e_\nu)$$

and the inner region

$$W_R = \bigcup_{\nu \in L} B_{R\varepsilon_n}(\operatorname{arctanh} z e_\nu).$$

We will choose r sufficiently small and R sufficiently large, while ε_n stems from Proposition 2.1. Remember that $\Psi = \Phi$ in the case $y = 0$ and that Φ attains a local maximum in $B_{c_2\varepsilon_n}(\operatorname{arctanh} z e_\nu)$ for each $\nu \in L$.

Outer region. First, we will show that the outer region does not contribute to the asymptotic behavior of (4.1). The estimate

$$\log \cosh x \leq \frac{1}{4\beta} x^2 + \max_{t \in \mathbb{R}} \left\{ -\frac{1}{4\beta} t^2 + \log \cosh t \right\} = \frac{1}{4\beta} x^2 + f_{\text{CW}}(2\beta)$$

and the bound on the random matrix given in Proposition 2.1 show that for \mathbb{P}_ξ -almost all ξ and all $n \geq n_0(\xi)$,

$$(4.2) \quad \Psi(\lambda) \leq -\frac{1}{2\beta} \|\lambda\|^2 + c_6 \|\lambda\| + \frac{1}{4\beta} \|\lambda\|^2 (1 + \varepsilon_n) + f_{\text{CW}}(2\beta)$$

with some constant $c_6 > 0$ depending on β , h and $\|y\|$ only. Therefore, provided that n is large enough to satisfy $\varepsilon_n \leq 1/3$, there exists a $\tau > 1$ such that for all λ satisfying $\|\lambda\| \geq \tau$, the right-hand side of (4.2) is bounded by $-\|\lambda\|^2/(8\beta)$. Let τ_p denote the radius which satisfies

$$(2\pi)^{-p/2} \int_{B_{\tau_p}(0)} \exp\left\{-\frac{1}{2}\|\lambda\|^2\right\} d\lambda = \frac{1}{2}.$$

By an isoperimetric inequality for Gaussian measures on \mathbb{R}^p (cf. [11], Theorem 1.2),

$$(4.3) \quad \left(\frac{n}{2\pi\beta} \right)^{p/2} \int_{B_{\tau_p}^c(0)} \exp\{n\Psi(\lambda)\} d\lambda \leq 2^{p-1} \exp\left\{-\frac{1}{2}\left[\sqrt{\frac{n}{4\beta}}\tau - \tau_p\right]^2\right\},$$

provided that n satisfies $\sqrt{n/(4\beta)}\tau > \tau_p$. Note that τ_p increases with the dimension p . Therefore, we want an upper bound for τ_p . Since

$$\begin{aligned} & (2\pi)^{-p/2} \int_{B_{\sqrt{2p}}^c(0)} \exp\left\{-\frac{1}{2}\|\lambda\|^2\right\} d\lambda \\ & \leq (2\pi)^{-p/2} \exp\left\{-\frac{p}{2}\right\} \int_{\mathbb{R}^p} \exp\left\{-\frac{1}{4}\|\lambda\|^2\right\} d\lambda = \left(\frac{2}{e}\right)^{p/2} \leq \frac{1}{2} \end{aligned}$$

for $p \geq 5$, it follows that

$$(4.4) \quad \tau_p \leq \max\{\sqrt{2p}, 4\}.$$

Going back to (4.3), we find

$$(4.5) \quad \left(\frac{n}{2\pi\beta}\right)^{p/2} \int_{B_\tau^c(0)} \exp\{n\Psi(\lambda)\} d\lambda \leq \exp\left\{-n\frac{\tau^2}{10\beta}\right\}$$

for large enough n . Next we will establish a lemma.

LEMMA 4.1. *For all $r > 0$,*

$$\limsup_{n \rightarrow \infty} \sup_{\lambda \in U_r \cap B_\tau(0)} \Psi(\lambda) < f_{\text{CW}}(\beta, h) \quad \mathbb{P}_\xi\text{-almost surely.}$$

This lemma together with (4.5) shows that for \mathbb{P}_ξ -almost all ξ , there exist a $\delta(\xi) > 0$ and an $n_3(\xi) \geq n_0(\xi)$ such that

$$\left(\frac{n}{2\pi\beta}\right)^{p/2} \int_{U_r} \exp\{n\Psi(\lambda)\} d\lambda \leq \exp\{n[f_{\text{CW}}(\beta, h) - \delta(\xi)]\}$$

for all $n \geq n_3(\xi)$. The proof of Lemma 4.1 is based on the following self-averaging property combined with a geometric argument.

LEMMA 4.2. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz continuous. Then for all $\tau > 0$ and \mathbb{P}_ξ -almost all ξ ,*

$$\lim_{n \rightarrow \infty} \sup_{\|v\| \leq \tau} \left| \frac{1}{n} \sum_{i=1}^n f(\langle v, \xi_i \rangle) - \mathbb{E}_\xi f(\langle v, \xi_1 \rangle) \right| = 0.$$

PROOF. Fix an arbitrary $\varepsilon \in (0, \tau)$ and choose $m = m(\varepsilon, \tau, p)$ points $v_1, \dots, v_m \in B_\tau(0)$ such that

$$\min_{j \in \{1, \dots, m\}} \|v - v_j\| \leq \varepsilon$$

for all $v \in \overline{B_\tau(0)}$. By selecting these points on a cubic lattice with spacing ε/\sqrt{p} , we may assume that $m \leq \exp\{p(\log(\tau/\varepsilon) + c)\}$; compare (3.6). For every $v \in \overline{B_\tau(0)}$ choose a $w(v) \in \{v_1, \dots, v_m\}$ with

$$\|v - w(v)\| = \min_{j \in \{1, \dots, m\}} \|v - v_j\|.$$

Denote by K_f the Lipschitz constant of f . It suffices to consider the case $K_f > 0$. Then for \mathbb{P}_ξ -almost all ξ , all $n \geq n_0(\xi)$ and all $v \in \overline{B_\tau(0)}$ by Proposition 2.1,

$$\begin{aligned} & \left| \frac{1}{n} \sum_{i=1}^n f(\langle v, \xi_i \rangle) - \frac{1}{n} \sum_{i=1}^n f(\langle w(v), \xi_i \rangle) \right| \\ & \leq K_f \left(\frac{1}{n} \sum_{i=1}^n \langle v - w(v), \xi_i \rangle^2 \right)^{1/2} \\ & \leq K_f (1 + \varepsilon_n)^{1/2} \|v - w(v)\| \leq K_f (1 + \varepsilon_n)^{1/2} \varepsilon. \end{aligned}$$

Furthermore, for all $v \in \overline{B_\tau(0)}$,

$$\begin{aligned} |\mathbb{E}_\xi f(\langle v, \xi_1 \rangle) - \mathbb{E}_\xi f(\langle w(v), \xi_1 \rangle)| & \leq K_f (\mathbb{E}_\xi (\langle v - w(v), \xi_1 \rangle^2))^{1/2} \\ & = K_f \|v - w(v)\| \leq K_f \varepsilon. \end{aligned}$$

Therefore, to prove the lemma it remains to show that

$$(4.6) \quad \lim_{n \rightarrow \infty} \sup_{j \in \{1, \dots, m\}} \left| \frac{1}{n} \sum_{i=1}^n f(\langle v_j, \xi_i \rangle) - \mathbb{E}_\xi f(\langle v_j, \xi_1 \rangle) \right| = 0$$

\mathbb{P}_ξ -almost surely. Recall that m depends on n via the dimension p . Fix j for the moment and let

$$\begin{aligned} A_{n,j} = & \left\{ (\tilde{\xi}_1, \dots, \tilde{\xi}_n) \in \{-1, 1\}^{pn} : \right. \\ & \left. \left| \frac{1}{n} \sum_{i=1}^n f(\langle v_j, \tilde{\xi}_i \rangle) - \mathbb{E}_\xi f(\langle v_j, \xi_1 \rangle) \right| < \tau K_f \varepsilon \right\} \end{aligned}$$

denote the set of those realizations of the patterns for which the difference we are interested in does not exceed the bound $\tau K_f \varepsilon$. Then,

$$\begin{aligned} (4.7) \quad & \mathbb{P}_\xi \left(\left| \frac{1}{n} \sum_{i=1}^n f(\langle v_j, \xi_i \rangle) - \mathbb{E}_\xi f(\langle v_j, \xi_1 \rangle) \right| \geq 2\tau K_f \varepsilon \right) \\ & \leq \mathbb{P}_\xi \left(\inf_{(\tilde{\xi}_1, \dots, \tilde{\xi}_n) \in A_{n,j}} \frac{1}{n} \sum_{i=1}^n |f(\langle v_j, \xi_i \rangle) - f(\langle v_j, \tilde{\xi}_i \rangle)| \geq \tau K_f \varepsilon \right) \end{aligned}$$

by the definition of $A_{n,j}$. Now,

$$\begin{aligned} (4.8) \quad & \inf_{(\tilde{\xi}_1, \dots, \tilde{\xi}_n) \in A_{n,j}} \frac{1}{n} \sum_{i=1}^n |f(\langle v_j, \xi_i \rangle) - f(\langle v_j, \tilde{\xi}_i \rangle)| \\ & \leq \frac{K_f}{n} \inf_{(\tilde{\xi}_1, \dots, \tilde{\xi}_n) \in A_{n,j}} \sum_{i=1}^n g_{v_j}(\xi_i, \tilde{\xi}_i), \end{aligned}$$

where $g_{v_j}(\xi_i, \tilde{\xi}_i) = |\langle v_j, \xi_i - \tilde{\xi}_i \rangle|$ serves as a weighted distance function on the set $\{-1, 1\}^p \times \{-1, 1\}^p$. By [13], Theorem 2.4.1, we have for all $t > 0$ the

isoperimetric inequality

$$(4.9) \quad \begin{aligned} \mathbb{E}_\xi \exp \left\{ t \inf_{(\tilde{\xi}_1, \dots, \tilde{\xi}_n) \in A_{n,j}} \sum_{i=1}^n g_{v_j}(\xi_i, \tilde{\xi}_i) \right\} \\ \leq \frac{1}{\mathbb{P}_\xi(A_{n,j})} (\mathbb{E}_\xi \cosh(tg_{v_j}(\xi_1, \xi_2)))^n. \end{aligned}$$

Calculating the expectation gives

$$\begin{aligned} \mathbb{E}_\xi \cosh(tg_{v_j}(\xi_1, \xi_2)) &= \mathbb{E}_\xi \exp\{t\langle v_j, \xi_1 - \xi_2 \rangle\} = \prod_{\mu=1}^p \left[\frac{1}{2} (1 + \cosh(2tv_j^\mu)) \right] \\ &\leq \prod_{\mu=1}^p \exp\{2t^2(v_j^\mu)^2\} = \exp\{2t^2\|v_j\|^2\} \leq \exp\{2t^2\tau^2\}, \end{aligned}$$

and by estimating the probability of $A_{n,j}^c$ with the help of Chebyshev's inequality, we get

$$\begin{aligned} \mathbb{P}_\xi(A_{n,j}^c) &\leq (n\tau K_f \varepsilon)^{-2} \mathbb{E}_\xi \left(\left| \sum_{i=1}^n (f(\langle v_j, \xi_i \rangle) - \mathbb{E}_\xi f(\langle v_j, \xi_1 \rangle)) \right|^2 \right) \\ &\leq (n\tau K_f \varepsilon)^{-2} \sum_{i=1}^n \mathbb{E}_\xi (|f(\langle v_j, \xi_i \rangle) - \mathbb{E}_\xi f(\langle v_j, \xi_1 \rangle)|^2) \\ &\leq n(n\tau K_f \varepsilon)^{-2} \mathbb{E}_\xi (|f(\langle v_j, \xi_1 \rangle) - f(0)|^2) \\ &\leq (n\tau^2 \varepsilon^2)^{-1} \mathbb{E}_\xi (\langle v_j, \xi_1 \rangle^2) \\ &\leq \frac{1}{n\varepsilon^2}. \end{aligned}$$

We will assume $n \geq 2/\varepsilon^2$ to assure $\mathbb{P}_\xi(A_{n,j}) \geq \frac{1}{2}$. By (4.7), (4.8), another application of Chebyshev's inequality and (4.9),

$$\mathbb{P}_\xi \left(\left| \frac{1}{n} \sum_{i=1}^n f(\langle v_j, \xi_i \rangle) - \mathbb{E}_\xi f(\langle v_j, \xi_1 \rangle) \right| \geq 2\tau K_f \varepsilon \right) \leq 2 \exp\{-n\varepsilon\tau t\} \exp\{2n\tau^2 t^2\}$$

for all $t > 0$. Choosing the optimal t , which is $t = \varepsilon/(4\tau)$, yields

$$\begin{aligned} \mathbb{P}_\xi \left(\sup_{j \in \{1, \dots, m\}} \left| \frac{1}{n} \sum_{i=1}^n f(\langle v_j, \xi_i \rangle) - \mathbb{E}_\xi f(\langle v_j, \xi_1 \rangle) \right| \geq 2\tau K_f \varepsilon \right) \\ \leq 2m \exp \left\{ -n \frac{\varepsilon^2}{8} \right\} \leq \exp \left\{ -n \frac{\varepsilon^2}{10} \right\} \end{aligned}$$

for all large enough n by the bound on m . Now (4.6) follows from the Borel–Cantelli lemma. \square

PROOF OF LEMMA 4.1. Recall the definitions of Ψ and y_n from Lemma 3.3. By the self-averaging property applied to the Lipschitz continuous function $\log \cosh$,

$$\lim_{n \rightarrow \infty} \sup_{\|\lambda\| \leq \tau} \left| \Psi(\lambda) + \frac{1}{2\beta} \|\lambda - he_l - y_n\|^2 - \mathbb{E}_\xi \log \cosh \langle \lambda, \xi_1 \rangle \right| = 0$$

\mathbb{P}_ξ -almost surely. Since

$$\begin{aligned} \left| \|\lambda - he_l - y_n\|^2 - \|\lambda - he_l\|^2 \right| &\leq 2|\langle \lambda - he_l, y_n \rangle| + \|y_n\|^2 \\ &\leq 2(\tau + h) \frac{\|y\|}{\sqrt{n}} + \frac{\|y\|^2}{n} \rightarrow 0 \end{aligned}$$

for all $\lambda \in B_\tau(0)$ and $\|\lambda - he_l\|^2 = \mathbb{E}_\xi(\langle \lambda - he_l, \xi_1 \rangle^2)$ for all $\lambda \in \mathbb{R}^p$, it suffices to show

$$(4.10) \quad \limsup_{n \rightarrow \infty} \sup_{\lambda \in U_r \cap B_\tau(0)} \mathbb{E}_\xi \left\{ -\frac{1}{2\beta} (\langle \lambda, \xi_1 \rangle - h\xi_1^l)^2 + \log \cosh \langle \lambda, \xi_1 \rangle \right\} < f_{\text{CW}}(\beta, h).$$

In the case $h = 0$, by the definition of z and $f_{\text{CW}}(\beta)$ this can be done by proving the existence of a constant $\delta > 0$, depending only on r , such that

$$(4.11) \quad \mathbb{P}_\xi(|\langle \lambda, \xi_1 \rangle| - |\operatorname{arctanh} z| \geq \delta) \geq \frac{1}{18}$$

for all $\lambda \in U_r \cap B_\tau(0)$ and all $p \in \mathbb{N}$. In the case $h \neq 0$, (4.10) can be established by showing

$$\mathbb{P}_\xi(|\langle \lambda, \xi_1 \rangle - \operatorname{arctanh} z \xi_1^l| \geq \delta) \geq \frac{1}{18}$$

for all $\lambda \in U_r \cap B_\tau(0)$ and all $p \in \mathbb{N}$ with a constant $\delta > 0$, depending only on r . We will show that $\delta = r/42$ suffices in both cases. The case $h \neq 0$ is a direct consequence of the following result on Rademacher averages, applied with $X = \langle \lambda - \operatorname{arctanh} z e_l, \xi_1 \rangle$ which satisfies $\mathbb{E}_\xi X^2 \geq r^2$. For background on Rademacher averages, see [11], Chapter 4, for instance.

LEMMA 4.3. *For every Rademacher average X ,*

$$\mathbb{P}(|X| \geq \tfrac{1}{8} \sqrt{\mathbb{E} X^2}) > \tfrac{1}{3}.$$

PROOF. By scaling we may assume that $\mathbb{E} X^2 = 1$. From the Khintchine inequality (cf. [11], Lemma 4.1), we know $\mathbb{E}|X| \geq 1/\sqrt{2}$. Therefore, by the Schwarz inequality,

$$\begin{aligned} \frac{1}{\sqrt{2}} &\leq \mathbb{E}|X| \leq \frac{1}{8} + \mathbb{E}(|X| 1_{\{|X| > 1/8\}}) \\ &\leq \frac{1}{8} + \sqrt{\mathbb{E}(X^2) \mathbb{P}\left(|X| > \frac{1}{8}\right)} \end{aligned}$$

and, therefore,

$$\mathbb{P}\left(|X| > \frac{1}{8}\right) \geq \left(\frac{1}{\sqrt{2}} - \frac{1}{8}\right)^2 > \frac{1}{3}. \quad \square$$

To treat the case $h = 0$, we fix $\lambda \in U_r \cap B_\tau(0)$. Without loss of generality, we may assume $z \geq 0$ as well as $|\lambda_1| \geq |\lambda_\mu|$ for all $\mu \in \{2, \dots, p\}$ and $\lambda_1 \geq 0$. In the sequel, we will use the abbreviations $\tilde{\lambda} = (\lambda_2, \dots, \lambda_p)$ and $\eta = (\xi_1^2, \dots, \xi_1^p)$. First, we will consider the case when $\|\tilde{\lambda}\|$ is small.

Let $\|\tilde{\lambda}\| \leq r/3$. Then $|\lambda_1 - \operatorname{arctanh} z|^2 = \|\lambda - \operatorname{arctanh} z e_1\|^2 - \|\tilde{\lambda}\|^2 \geq 8r^2/9$. From the triangle inequality, we find $||x + y| - \tilde{z}| \geq ||x| - \tilde{z}| - |y|$ for all $x, y \in \mathbb{R}$ and all $\tilde{z} \geq 0$. Therefore, by Chebyshev's inequality,

$$\begin{aligned} \mathbb{P}_\xi\left(|\langle \lambda, \xi_1 \rangle| - \operatorname{arctanh} z \geq \frac{r}{3}\right) &= \mathbb{P}_\xi\left(|\lambda_1 \xi_1^1 + \langle \tilde{\lambda}, \eta \rangle| - \operatorname{arctanh} z \geq \frac{r}{3}\right) \\ &\geq \mathbb{P}_\xi\left(|\lambda_1 - \operatorname{arctanh} z| - |\langle \tilde{\lambda}, \eta \rangle| \geq \frac{r}{3}\right) \\ &\geq 1 - \mathbb{P}_\xi\left(|\langle \tilde{\lambda}, \eta \rangle| > \frac{r}{2}\right) \\ &\geq 1 - \left(\frac{2}{r}\right)^2 \mathbb{E}_\xi(\langle \tilde{\lambda}, \eta \rangle^2) \geq \frac{5}{9}. \end{aligned}$$

It remains to consider the case $\|\tilde{\lambda}\| > r/3$. When $|\lambda_1| \leq r/\sqrt{27}$, then there exists a $k \in \{2, \dots, p-2\}$ such that

$$a = \left(\sum_{\mu=1}^k \lambda_\mu^2\right)^{1/2} \geq \frac{r}{\sqrt{27}} \quad \text{and} \quad b = \left(\sum_{\mu=k+1}^p \lambda_\mu^2\right)^{1/2} \geq \frac{r}{\sqrt{27}}.$$

(Recall that $\lambda_1 \geq |\lambda_\mu|$ for all $\mu \in \{2, \dots, p\}$ by assumption.) When $|\lambda_1| > r/\sqrt{27}$, then the same holds with $k = 1$. Let

$$X = \frac{1}{a} \sum_{\mu=1}^k \lambda_\mu \xi_1^\mu \quad \text{and} \quad Y = \frac{1}{b} \sum_{\mu=k+1}^p \lambda_\mu \xi_1^\mu.$$

Then $\langle \lambda, \xi_1 \rangle$ can be rewritten as $\langle \lambda, \xi_1 \rangle = aX + bY$ with $a, b \geq r/\sqrt{27}$ and independent Rademacher averages X and Y , satisfying $\mathbb{E}_\xi X^2 = \mathbb{E}_\xi Y^2 = 1$. Now, (4.11) follows from the following lemma with $\varrho = r/\sqrt{27}$, and the proof of Lemma 4.1 is complete. \square

LEMMA 4.4. *Let $\varrho > 0$, $a, b \geq \varrho$ and $\tilde{z} \geq 0$. Consider independent Rademacher averages X and Y , satisfying $\mathbb{E} X^2 = \mathbb{E} Y^2 = 1$. Then*

$$\mathbb{P}(|aX + bY| - \tilde{z}| \geq \varrho/8) > 1/18.$$

PROOF. Let $\kappa, \tilde{\kappa}$ be two independent Bernoulli random variables, independent of X and Y , satisfying $\mathbb{P}(\kappa = \pm 1) = \mathbb{P}(\tilde{\kappa} = \pm 1) = 1/2$. Then $aX + bY$ equals $a\kappa X + b\tilde{\kappa}Y$ in distribution and, therefore,

$$(4.12) \quad \mathbb{P}\left(|aX + bY| - \tilde{z} \geq \frac{\varrho}{8}\right) = \mathbb{E}\left(\mathbb{P}\left(|a\kappa X + b\tilde{\kappa}Y| - \tilde{z} \geq \frac{\varrho}{8} \middle| X, Y\right)\right).$$

For the conditional probability we have the estimate

$$\mathbb{P}\left(|a\kappa X + b\tilde{\kappa}Y| - \tilde{z} \geq \frac{\varrho}{8} \middle| X, Y\right) \geq \frac{1}{2} \mathbf{1}_{\{\min\{|aX|, |bY|\} \geq \varrho/8\}},$$

which can be seen as follows: assume $\min\{|aX|, |bY|\} \geq \varrho/8$ and suppose that $|aX + bY| - \tilde{z} < \varrho/8$. Then the triangle inequality implies

$$|aX - bY| - \tilde{z} \geq |aX - bY| - |aX + bY| - \frac{\varrho}{8} = 2 \min\{|aX|, |bY|\} - \frac{\varrho}{8} \geq \frac{\varrho}{8}.$$

Therefore, at least half of the realizations of the signs κ and $\tilde{\kappa}$ lead to the estimate $|a\kappa X + b\tilde{\kappa}Y| - \tilde{z} \geq \varrho/8$. Returning to (4.12), we find

$$\mathbb{P}\left(|aX + bY| - \tilde{z} \geq \frac{\varrho}{8}\right) \geq \frac{1}{2} \mathbb{P}\left(\min\{|aX|, |bY|\} \geq \frac{\varrho}{8}\right).$$

By Lemma 4.3, we know $\mathbb{P}(a|X| \geq \varrho/8) > 1/3$ and $\mathbb{P}(b|Y| \geq \varrho/8) > 1/3$. Therefore,

$$\mathbb{P}\left(|aX + bY| - \tilde{z} \geq \frac{\varrho}{8}\right) > \frac{1}{18}.$$

□

Taylor expansion of Ψ . Before turning to the intermediate and the inner region, we need to investigate Ψ given in (3.19). Since $\lambda_\nu^n(\xi)$ for $\nu \in L$ is a local maximum of Φ by Proposition 2.3(b), the Taylor expansion of Ψ in $\lambda_\nu^n(\xi)$ gives

$$(4.13) \quad \begin{aligned} \Psi(\lambda) &= \Phi(\lambda_\nu^n(\xi)) + \langle x_\nu^n(\xi), y_n \rangle - \frac{1}{2\beta} \|y_n\|^2 + \frac{1}{\beta} \langle \lambda - \lambda_\nu^n(\xi), y_n \rangle \\ &\quad - \frac{1}{2} \langle \lambda - \lambda_\nu^n(\xi), C_\nu^n(\xi)[\lambda - \lambda_\nu^n(\xi)] \rangle + R_\nu^n(\lambda - \lambda_\nu^n(\xi), \xi) \end{aligned}$$

for all $\lambda \in \mathbb{R}^p$ and all $n \geq n_1(\xi)$, where $C_\nu^n(\xi)$ is given by (3.18). The remainder R_ν^n is given by

$$R_\nu^n(\lambda, \xi) = -\frac{1}{n} \sum_{i=1}^n \left(\int_0^1 (1-t)^2 \frac{\tanh}{\cosh^2}(\langle \lambda_\nu^n(\xi) + t\lambda, \xi_i \rangle) dt \right) \langle \lambda, \xi_i \rangle^3.$$

By Proposition 2.1 and Lemma 3.1, we have the following bound on the remainder: for arbitrary $a, \varrho > 0$, both possibly depending on $n \in \mathbb{N}$, for \mathbb{P}_ξ -almost

all ξ , all $n \geq n_2(\xi)$ and all $\lambda \in B_\varrho(0)$,

$$\begin{aligned}
 |R_\nu^n(\lambda, \xi)| &\leq \frac{1}{n} \sum_{i=1}^n \langle \lambda, \xi_i \rangle^2 \mathbf{1}_{\{|\langle \lambda, \xi_i \rangle| > a\}} \left| \int_{\langle \lambda_\nu^n(\xi), \xi_i \rangle}^{\langle \lambda_\nu^n(\xi) + \lambda, \xi_i \rangle} \frac{|\tanh|}{\cosh^2}(s) ds \right| \\
 &\quad + \frac{1}{n} \sum_{i=1}^n |\langle \lambda, \xi_i \rangle|^3 \mathbf{1}_{\{|\langle \lambda, \xi_i \rangle| \leq a\}} \int_0^1 \frac{|\tanh|}{\cosh^2}(\langle \lambda_\nu^n(\xi) + t\lambda, \xi_i \rangle) dt \\
 (4.14) \quad &\leq \Gamma\left(n, \frac{a}{\varrho}\right) \|\lambda\|^2 4 \int_{\mathbb{R}} \exp(-2|s|) ds + a \frac{1}{n} \sum_{i=1}^n \langle \lambda, \xi_i \rangle^2 \\
 &\leq \left[4\Gamma\left(n, \frac{a}{\varrho}\right) + a(1 + \varepsilon_n) \right] \|\lambda\|^2.
 \end{aligned}$$

In the case $\beta < \beta_c$ and $h = 0$, we know from Proposition 2.3(b) that $\lambda_\nu^n(\xi) = 0$ and can improve this bound to

$$(4.15) \quad |R_\nu^n(\lambda, \xi)| \leq [4\Gamma(n, a/\varrho) + a^2(1 + \varepsilon_n)] \|\lambda\|^2$$

by the estimate

$$\frac{|\tanh|}{\cosh^2}(\langle t\lambda, \xi_i \rangle) \leq |\langle \lambda, \xi_i \rangle|$$

for all $t \in [0, 1]$.

Intermediate region. Let $a \leq c_1/16$ with c_1 from (2.5) and let r be small enough and $n \geq \max\{n_1(\xi), n_2(\xi)\}$ large enough to satisfy

$$4\Gamma\left(n, \frac{a}{r}\right) + a(1 + \varepsilon_n) \leq \frac{c_1}{8} \quad \text{and} \quad \varepsilon_n \leq 1.$$

Furthermore, let

$$R > \max\left\{ 2\left[c_2 + \frac{\|y\|}{c_1\beta} \right], \frac{5}{\sqrt{c_1}} \right\}$$

with $c_2 = 2|z|/c_1$ as before. By the Taylor expansion (4.13), for every $\nu \in L$,

$$\begin{aligned}
 &\left(\frac{n}{2\pi\beta} \right)^{p/2} \int_{\{R\varepsilon_n \leq \|\lambda - \operatorname{arctanh} z e_\nu\| < r\}} \exp\{n\Psi(\lambda)\} d\lambda \\
 (4.16) \quad &= \left(\frac{n}{2\pi\beta} \right)^{p/2} \exp\left\{ n \left[\Phi(\lambda_\nu^n(\xi)) + \langle x_\nu^n(\xi), y_n \rangle - \frac{1}{2\beta} \|y_n\|^2 \right] \right\} \\
 &\quad \times \int_{\{R\varepsilon_n \leq \|\lambda - \operatorname{arctanh} z e_\nu\| < r\}} \exp\left\{ -\frac{n}{2} \langle \lambda - \lambda_\nu^n(\xi), C_\nu^n(\xi) [\lambda - \lambda_\nu^n(\xi)] \rangle \right\} \\
 &\quad \times \exp\left\{ \frac{n}{\beta} \langle \lambda - \lambda_\nu^n(\xi), y_n \rangle + nR_\nu^n(\lambda - \lambda_\nu^n(\xi), \xi) \right\} d\lambda.
 \end{aligned}$$

Before estimating the right-hand side of (4.16), let us remark that the following estimates will hold uniformly for $\nu \in L$. Substituting $\lambda + \lambda_\nu^n(\xi) +$

$(1/\beta)C_\nu^n(\xi)^{-1}y_n$ for λ and applying the bound (4.14) on the remainder, we see that the integral on the right-hand side of (4.16) is bounded above by

$$(4.17) \quad \exp\left\{\frac{n}{2\beta^2}\langle y_n, C_\nu^n(\xi)^{-1}y_n \rangle\right\} \\ \times \int \exp\left\{-\frac{n}{2}\langle \lambda, C_\nu^n(\xi)\lambda \rangle + n\frac{c_1}{8}\left\|\lambda + \frac{1}{\beta}C_\nu^n(\xi)^{-1}y_n\right\|^2\right\} d\lambda,$$

where the domain of integration

$$\left\{\left\|\lambda + \lambda_\nu^n(\xi) - \operatorname{arctanh} z e_\nu + \frac{1}{\beta}C_\nu^n(\xi)^{-1}y_n\right\| \geq R\varepsilon_n\right\}$$

is contained in $\{\|\lambda\| \geq R\varepsilon_n/2\}$, because $\|\lambda_\nu^n(\xi) - \operatorname{arctanh} z e_\nu\| \leq c_2\varepsilon_n$ and

$$(4.18) \quad \left\|\frac{1}{\beta}C_\nu^n(\xi)^{-1}y_n\right\| \leq \frac{\|y\|}{c_1\beta\sqrt{n}}$$

by Proposition 2.3(a). Again from Proposition 2.3(a), we know that

$$\left\|C_\nu^n(\xi) - \frac{c_1}{2}\operatorname{Id}_{\mathbb{R}^p}\right\| \geq \frac{c_1}{2}.$$

Substituting λ for $\sqrt{n}(C_\nu^n(\xi) - (c_1/2)\operatorname{Id}_{\mathbb{R}^p})^{1/2}\lambda$, we find that the integral in (4.17) is bounded above by

$$\frac{1}{n^{p/2}} \exp\left\{\frac{\|y\|^2}{4c_1\beta^2}\right\} \det\left(C_\nu^n(\xi) - \frac{c_1}{2}\operatorname{Id}_{\mathbb{R}^p}\right)^{-1/2} \\ \times \int_{\{\|\lambda\| \geq \sqrt{c_1/2}(R/2)\sqrt{n}\varepsilon_n\}} \exp\left\{-\frac{1}{2}\|\lambda\|^2\right\} d\lambda.$$

Another application of Theorem 1.2 in [11] and (4.4) shows that the last integral is bounded by

$$(4.19) \quad \frac{1}{2}(2\pi)^{p/2} \exp\left\{-\frac{1}{2}\left[\sqrt{\frac{c_1}{2}}\frac{R}{2}\sqrt{n}\varepsilon_n - \tau_p\right]^2\right\} \\ \leq \frac{1}{2}(2\pi)^{p/2} \exp\left\{-\frac{1}{2}\left[\sqrt{\frac{c_1}{2}}\frac{R}{2}\sqrt{n}\varepsilon_n\right]^2\right\},$$

where we used the fact that $R\sqrt{c_1} \geq 5$ implies

$$\sqrt{\frac{c_1}{2}}\frac{R}{2}\sqrt{n}\varepsilon_n \geq 2\tau_p$$

for large enough n .

Putting the above estimates together, we find that

$$\begin{aligned} & \left(\frac{n}{2\pi\beta}\right)^{p/2} \int_{\{R\varepsilon_n \leq \|\lambda - \operatorname{arctanh} z e_\nu\| < r\}} \exp\{n\Psi(\lambda)\} d\lambda \\ & \leq \frac{1}{2} \exp\left\{\frac{\|y\|^2}{4c_1\beta^2}\right\} \exp\left\{-\frac{c_1 R^2}{64} n \varepsilon_n^2\right\} \beta^{-p/2} \det\left(C_\nu^n(\xi) - \frac{c_1}{2} \operatorname{Id}_{\mathbb{R}^p}\right)^{-1/2} \\ & \quad \times \exp\left\{n\left[\Phi(\lambda_\nu^n(\xi)) + \langle x_\nu^n(\xi), y_n \rangle - \frac{1}{2\beta} \|y_n\|^2 + \frac{1}{2\beta^2} \langle y_n, C_\nu^n(\xi)^{-1} y_n \rangle\right]\right\}. \end{aligned}$$

By Proposition 2.3(a), the smallest eigenvalue of $C_\nu^n(\xi)$ is greater than or equal to c_1 . Hence

$$\det\left(C_\nu^n(\xi) - \frac{c_1}{2} \operatorname{Id}_{\mathbb{R}^p}\right) \geq \det\left(\frac{1}{2} C_\nu^n(\xi)\right) = 2^{-p} \det(C_\nu^n(\xi)).$$

Since $c_1 R^2 \geq 25$ by the choice of R , we know that

$$2^{p/2-1} \exp\left\{-\frac{c_1 R^2}{64} n \varepsilon_n^2\right\} \exp\left\{\frac{\|y\|^2}{4c_1\beta^2}\right\} = o(1),$$

where the error term $o(1)$ does not depend on ν . Therefore,

$$\begin{aligned} & \left(\frac{n}{2\pi\beta}\right)^{p/2} \int_{\{R\varepsilon_n \leq \|\lambda - \operatorname{arctanh} z e_\nu\| < r\}} \exp\{n\Psi(\lambda)\} d\lambda \\ & \leq o(1) \det(\beta C_\nu^n(\xi))^{-1/2} \exp\{n[\Phi(\lambda_\nu^n(\xi)) + \langle x_\nu^n(\xi), y_n \rangle]\} \\ & \quad \times \exp\left\{-\frac{n}{2\beta} \left\langle y_n, \left[\operatorname{Id}_{\mathbb{R}^p} - \frac{1}{\beta} C_\nu^n(\xi)^{-1}\right] y_n \right\rangle\right\} \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{n}{2\pi\beta}\right)^{p/2} \int_{V_{r,R}} \exp\{n\Psi(\lambda)\} d\lambda \\ & \leq \left(\sum_{\nu \in L} \det(\beta C_\nu^n(\xi))^{-1/2} \exp\{n[\Phi(\lambda_\nu^n(\xi)) + \langle x_\nu^n(\xi), y_n \rangle]\}\right) \\ & \quad \times \exp\left\{-\frac{n}{2\beta} \left\langle y_n, \left[\operatorname{Id}_{\mathbb{R}^p} - \frac{1}{\beta} C_\nu^n(\xi)^{-1}\right] y_n \right\rangle\right\} o(1). \end{aligned}$$

This shows that the intermediate region does not contribute to the asymptotic behavior.

Inner region. First of all, let us remark that the following estimates will hold uniformly for $\nu \in L$ as the corresponding ones did in the previous case of the intermediate region. Choose R as before. We will apply the bounds on the remainder of the Taylor expansion with a replaced by

$$a_n = \begin{cases} 3R\varepsilon_n \sqrt{\log p}, & \text{for unbounded } p, \\ \sqrt{\varepsilon_n}, & \text{otherwise.} \end{cases}$$

As an abbreviation, set

$$\delta_n = \begin{cases} 4\Gamma\left(n, \frac{a_n}{R\varepsilon_n}\right) + a_n(1 + \varepsilon_n), & \text{in the case } z \neq 0, \\ 4\Gamma\left(n, \frac{a_n}{R\varepsilon_n}\right) + a_n^2(1 + \varepsilon_n), & \text{otherwise.} \end{cases}$$

Let $n \geq \max\{n_1(\xi), n_2(\xi)\}$ be large enough to satisfy $8\delta_n \leq c_1$ which is possible since $\delta_n \rightarrow 0$. Note that $p\delta_n$ tends to zero in the following two cases: first, $z = 0$ and $(p^2 \log p)/n \rightarrow 0$ imply $p\delta_n \rightarrow 0$; second, $(p^3 \log p)/n \rightarrow 0$ always implies $p\delta_n \rightarrow 0$.

As in the treatment of the intermediate region, by the Taylor expansion (4.13),

$$\begin{aligned} & \left(\frac{n}{2\pi\beta}\right)^{p/2} \int_{\{\|\lambda - \operatorname{arctanh} z e_\nu\| < R\varepsilon_n\}} \exp\{n\Psi(\lambda)\} d\lambda \\ (4.20) \quad &= \left(\frac{n}{2\pi\beta}\right)^{p/2} \exp\left\{n\left[\Phi(\lambda_\nu^n(\xi)) + \langle x_\nu^n(\xi), y_n \rangle - \frac{1}{2\beta}\|y_n\|^2\right]\right\} \\ &\quad \times \int_{\{\|\lambda - \operatorname{arctanh} z e_\nu\| < R\varepsilon_n\}} \exp\left\{-\frac{n}{2}\langle \lambda - \lambda_\nu^n(\xi), C_\nu^n(\xi)[\lambda - \lambda_\nu^n(\xi)] \rangle\right\} \\ &\quad \times \exp\left\{\frac{n}{\beta}\langle \lambda - \lambda_\nu^n(\xi), y_n \rangle + nR_\nu^n(\lambda - \lambda_\nu^n(\xi), \xi)\right\} d\lambda. \end{aligned}$$

Again substituting $\lambda + \lambda_\nu^n(\xi) + (1/\beta)C_\nu^n(\xi)^{-1}y_n$ for λ and applying the bound (4.14), respectively, (4.15) on the remainder, we get the following upper, respectively, lower bound for the integral on the right-hand side of (4.20):

$$\begin{aligned} (4.21) \quad & \exp\left\{\frac{n}{2\beta^2}\langle y_n, C_\nu^n(\xi)^{-1}y_n \rangle\right\} \\ & \times \int \exp\left\{-\frac{n}{2}\langle \lambda, C_\nu^n(\xi)\lambda \rangle \pm n\delta_n \left\|\lambda + \frac{1}{\beta}C_\nu^n(\xi)^{-1}y_n\right\|^2\right\} d\lambda, \end{aligned}$$

where the corresponding domain of integration is

$$\left\{\left\|\lambda + \lambda_\nu^n(\xi) - \operatorname{arctanh} z e_\nu + \frac{1}{\beta}C_\nu^n(\xi)^{-1}y_n\right\| < R\varepsilon_n\right\}.$$

By the choice of R , it contains $\{\|\lambda\| < R\varepsilon_n/2\}$. Replacing $\sqrt{n}(C_\nu^n(\xi) \mp 4\delta_n \operatorname{Id}_{\mathbb{R}^p})^{1/2}\lambda$ by λ shows that the integral in (4.21) is bounded above, respectively, below by

$$\begin{aligned} (4.22) \quad & \frac{1}{n^{p/2}} \exp\left\{\pm 2n\delta_n \left\|\frac{1}{\beta}C_\nu^n(\xi)^{-1}y_n\right\|^2\right\} \det(C_\nu^n(\xi) \mp 4\delta_n \operatorname{Id}_{\mathbb{R}^p})^{-1/2} \\ & \times \left[\int_{\mathbb{R}^p} \exp\left\{-\frac{1}{2}\|\lambda\|^2\right\} d\lambda \pm \int_{\{\|\lambda\| \geq \sqrt{c_1/2}(R/2)\sqrt{n\varepsilon_n}\}} \exp\left\{-\frac{1}{2}\|\lambda\|^2\right\} d\lambda\right]. \end{aligned}$$

By (4.18) and the bound (4.19) on the second integral in (4.22), for large enough n the expression (4.22) is bounded above, respectively, below by

$$e^{o(1)} \left(\frac{2\pi}{n} \right)^{p/2} (\det(C_\nu^n(\xi)) \det(\text{Id}_{\mathbb{R}^p} \mp 4\delta_n C_\nu^n(\xi)^{-1}))^{-1/2} \left[1 \pm \frac{1}{2} \exp \left[-\frac{c_1 R^2}{64} n \varepsilon_n^2 \right] \right].$$

Since the smallest eigenvalue of $C_\nu^n(\xi)$ is bounded below by c_1 ,

$$\left(1 - \frac{4\delta_n}{c_1} \right)^p \leq \det(\text{Id}_{\mathbb{R}^p} \pm 4\delta_n C_\nu^n(\xi)^{-1}) \leq \left(1 + \frac{4\delta_n}{c_1} \right)^p.$$

Hence,

$$\begin{aligned} & \left(\frac{n}{2\pi\beta} \right)^{p/2} \int_{W_R} \exp\{n\Psi(\lambda)\} d\lambda \\ &= \left(\sum_{\nu \in L} \det(\beta C_\nu^n(\xi))^{-1/2} \exp\{n[\Phi(\lambda_\nu^n(\xi)) + \langle x_\nu^n(\xi), y_n \rangle] \} \right. \\ & \quad \times \exp \left\{ -\frac{n}{2\beta} \left\langle y_n, \left[\text{Id}_{\mathbb{R}^p} - \frac{1}{\beta} C_\nu^n(\xi)^{-1} \right] y_n \right\rangle \right\} \Bigg) \\ & \quad \times [1 + o(1)] \exp\{\mathcal{O}(p\delta_n)\}, \end{aligned}$$

and the proof of Lemma 3.3 is complete. \square

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